

## Noncommutative Fluids

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**Abstract.** We review the connection between noncommutative gauge theory, matrix models and fluid mechanical systems. The noncommutative Chern-Simons description of the quantum Hall effect and bosonization of collective fermion states are used as specific examples.

*This is a draft version of the final writeup, for distribution only to the audience of the corresponding lecture at the Poincaré Institute. It is incomplete in various ways and likely to contain mistakes, omissions or inaccuracies. Citations, in particular, are exceedingly preliminary and by no means represent the intentions of the author on what to include or omit in the final version.*

### 1 Introduction

The idea that space may be a derived or emergent concept is a relatively old theme in theoretical physics. In the context of quantum mechanics, observables are operators and it is only their spectrum and mutual relations (commutators) that define their physical content. Space, to the extent that it is observable, should be no different. The properties attributed to space from everyday experience -and postulated in newtonian mechanics and special relativity- could be either exact or approximate, emerging in some particular or partial classical limit. Other structures, extending or deforming the concepts of classical geometry, and reducing to it under appropriate conditions, are conceivable.

This possibility has had an early emergence in speculations by Heisenberg himself. It made reappearances in various guises and contexts [1]. One of the most strikingly prescient of later developments in noncommutative gauge theory was the work of Eguchi and Kawai in large- $N$  single-plaquette lattice gauge theory [2]. It was, however, after the seminal and celebrated work of Alain Connes that noncommutative geometry achieved the mathematical rigor and conceptual richness that made it a major component of modern theoretical physics. The concept made further inroads when it emerged as a property of spacetime solutions derived from string theory [4, 5] and, by now, it claims a huge body of research literature.

One of the reasons that makes the idea of noncommutative spaces attractive is the common language and connections that it provides between apparently disparate topics. Indeed, as will be reviewed in this writeup, noncommutative physics unifies such *a priori* different objects as gauge fields, membranes, fluids, matrix models and many-body systems. (Some of the above connections can be established independently, but the full continuum emerges only in the noncommutative setting.)

Unification of description usually brings unification of concepts. This raises the stakes and elevates noncommutativity into a possibly fundamental property of na-

ture. We could ask, for instance, whether the eventual bringing together of gravity, quantum mechanics and thermodynamics will arise out of some underlying fully non-commutative structure that shapes into spacetime, quantum mechanics and statistical ensembles in some appropriate limit. Whether this is indeed true is, of course, unclear and leaves room for wild speculation.

At this point, we should refrain from fantasizing any further and take a more pragmatic point of view. The obvious question is: does noncommutativity buy us any advantage for physics as we presently know it? It will be the purpose of this exposé (as, I imagine, of the other talks in this session of the Poincaré Institute) to demonstrate that this, indeed, is the case.

## 2 Review of noncommutative spaces

The concepts of noncommutative geometry will be covered by other speakers in this session and there is probably little use in repeating them here. Moreover, there are many excellent and complete review articles, of which [6, 7, 8] are only a small sample.

Nevertheless, a brief summary will be presented here, for two main reasons. Firstly, it will make this writeup essentially self-contained and will minimize the need to refer to other sources for a coherent reading; and secondly, the level and tone of the presentation will be adapted to our needs, and hopefully will serve as a low-key alternative to more rigorous and complete treatments.

### 2.1 The operator formulation

The simplest starting point for the definition of noncommutative spaces is through the definition of noncommutative coordinates. This is the approach that is most closely related to physics, making the allusions to quantum mechanics most explicit, and is therefore also the most common one in physics texts. In this, the noncommutative spaces are defined in terms of their coordinates  $X^\mu$ , which are abstracted into (linear) operators. Such coordinates can be added and multiplied (associatively), forming a full operator algebra, but are not (necessarily) commutative. Instead, they obey the commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad \mu, \nu = 1, \dots, d \quad (1)$$

The antisymmetric two-tensor  $\theta^{\mu\nu}$  could be itself an operator, but is usually taken to commute with all  $x^\mu$  (for ‘flat’ noncommutative spaces) and is, thus, a set of ordinary, constant c-numbers. Its inverse, when it exists,

$$\omega_{\mu\nu} = (\theta^{-1})_{\mu\nu} \quad (2)$$

defines a constant two-form  $\omega$  characterizing the noncommutativity of the space.

Clearly the form of  $\theta$  can be changed by redefining the coordinates of the space. Linear redefinitions of the  $x^\mu$ , in particular, would leave  $\theta^{\mu\nu}$  a c-number (nonlinear redefinitions will be examined later). We can take advantage of this to give a simple form to  $\theta^{\mu\nu}$ . Specifically, by an orthogonal transformation of the  $x^\mu$  we can bring  $\theta^{\mu\nu}$  to a Darboux form consisting of two-dimensional blocks proportional to  $i\sigma_2$  plus a set of zero eigenvalues. This would decompose the space into a direct sum of mutually commuting two-dimensional noncommutative subspaces, plus possibly a number of commuting coordinates (odd-dimensional spaces necessarily have at least one commuting coordinate). In general, there will be  $2n$  properly noncommuting coordinates

$x^\alpha$  ( $\alpha = 1, \dots, 2n$ ) and  $q = d - 2n$  commuting ones  $Y^i$  ( $i = 1, \dots, q$ ). In that case  $\omega$  will be defined as the inverse of the projection  $\bar{\theta}$  of  $\theta$  on the fully noncommuting subspace:

$$\omega_{\alpha\beta} = (\bar{\theta}^{-1})_{\alpha\beta}, \quad \omega_{ij} = 0 \quad (3)$$

The actual noncommutative space can be thought of as a representation of the above operator algebra (1), acting on a set of states. For real spaces the operators  $x^\mu$  will be considered hermitian, their eigenvalues corresponding to possible values of the corresponding coordinate. Not all coordinates can be diagonalized simultaneously, so the notion of ‘points’ (sets of values for all coordinates  $x^\mu$ ) is absent. The analogy with quantum mechanical coordinate and momentum is clear, with each ‘Darboux’ pair of noncommutative coordinates being the analog of a canonical quantum pair. Nevertheless, a full set of geometric notions survives, in particular relating to fields on the space, as will become clear.

The representation of  $x^\mu$  can be reducible or irreducible. For the commuting components  $Y^i$  any useful representation must necessarily be reducible, else the corresponding directions would effectively be absent (consisting of a single point). States are labeled by the values of these coordinates  $y^i$ , taken to be continuous. The rest of the space, consisting of canonical Heisenberg pairs, admits the tensor product of Heisenberg-Fock Hilbert spaces (one for each two-dimensional noncommuting subspace  $k = 1, \dots, n$ ) as its unique irreducible representation. In general, we can have a reducible representation consisting of the direct sum of  $N$  such irreducible components for each set of values  $y^i$ , labeled by an extra index  $a = 1, \dots, N$  (we shall take  $N$  not to depend on  $y^i$ ). A complete basis for the states, then, can be

$$|n_1, \dots, n_n; y^1, \dots, y^q; a\rangle \quad (4)$$

where  $n_k$  is the Fock (oscillator) excitation number of the  $k$ -th two-dimensional subspace.

Due to the reducibility of the above representation, the operators  $x^\mu$  do not constitute a complete set. To make the set complete, additional operators need be introduced. To deal with the reducibility due to the values  $y^i$ , we consider translation (derivative) operators  $\partial_\mu$ . These are defined through their action on  $x^\mu$ , generating constant shifts:

$$[\partial_\mu, x^\nu] = \delta_\mu^\nu \quad (5)$$

On the fully noncommutative subspace these are inner automorphisms generated by

$$\partial_\alpha = -i\omega_{\alpha\beta}x^\beta \quad (6)$$

For the commutative coordinates, however, extra operators have to be appended, shifting the Casimirs  $Y^i$  and thus acting on the coordinates  $y^i$  as usual derivatives.

To deal with the reducibility due to the components  $a = 1, \dots, N$ , we need to introduce yet another set of operators in the full representation space mixing the above  $N$  components. Such a set are the hermitian  $U(N)$  operators  $G^r$ ,  $r = 1, \dots, N^2$  that commute with the  $x^\mu$ ,  $\partial_\mu$  and mix the components  $a$ . (We could, of course, choose these operators to be the  $SU(N)$  subset, eliminating the trivial identity operator.) The set of operators  $x^\alpha, \partial_i, G^r$  is now complete.

Within the above setting, we can define field theories on a noncommutative space. Fields are the analogs of functions of coordinates  $x^\mu$ ; that is, arbitrary operators in the universal enveloping algebra of the  $x^\mu$ . In general, the above fields are *not* arbitrary operators on the full representation space, since they commute with  $\partial_i$  and

$G^r$ . In particular, they act ‘pointwise’ on the commutative coordinates  $Y^i$  are are, therefore, ordinary functions of the  $y^i$ .

We can, of course, define fields depending also on the remaining operators. Fields involving operators  $G^r$  are useful, as they act as  $N \times N$  matrices on components  $a$ . They are the analogs of matrix-valued fields and will be useful in constructing gauge theories. We could further define operators that depend on the commutative derivatives  $\partial_i$ . These have no commutative analog, and will not be considered here. Notice, however, that on fully noncommutative spaces (even-dimensional spaces without commutative components), the matrix-valued fields  $f^{ab}(x^\mu)$  constitute the *full* set of operators acting on the representation space.

The fundamental notions completing the discussion of noncommutative field theory are the definitions of derivatives and space integral. Derivatives of a function  $f$  are defined as commutators with the corresponding operator:

$$\partial_\mu \cdot f = [\partial_\mu, f] \quad (7)$$

That is, through the adjoint action of the operator  $\partial_\mu$  on fields (we use the dot to denote this action). For the commutative derivatives  $\partial_i$  this is the ordinary partial derivative  $\partial/\partial y^i$ . For the noncommutative coordinates, however, such action is generated by the  $x^\alpha$  themselves, as  $\partial_\alpha = -i\omega_{\alpha\beta}x^\beta$ . So the notion of coordinates and derivatives on purely noncommutative spaces fuses, the distinction made only upon specifying the action of the operators  $x^\alpha$  on fields (left- or right- multiplication, or adjoint action).

The integral over space is defined as the trace in the representation space, normalized as:

$$\int d^d x = \int d^q y \operatorname{tr}' \sqrt{\det(2\pi\theta)} \operatorname{tr} \equiv \operatorname{Tr} \quad (8)$$

where  $\operatorname{tr}$  is the trace over the Fock spaces and  $\operatorname{tr}'$  is the trace over the degeneracy index  $a = 1, \dots, N$ . This corresponds to a space integral and a trace over the matrix indices  $a$ . The extra determinant factor ensures the recovery of the proper commutative limit (think of semiclassical quantization, or the transition from quantum to classical statistical mechanical partition functions.)

All manipulations with ordinary field theory can be transposed here, with a noncommutative twist. For instance, the fact that the integral of a total derivative vanishes (under proper boundary conditions), translates to the statement that the trace of a commutator vanishes, and its violation by fields with nontrivial behavior at infinity is mirrored in the nonvanishing trace of the commutator of unbounded, non-trace class operators, such as the noncommutative coordinates themselves.

## 2.2 Weyl maps, Wigner functions and \*-products

The product of noncommutative fields is simply the product of the corresponding operators, which is clearly associative but not commutative. It is also not ‘pointwise’, as the notion of points does not even exist. Nevertheless, in the limit  $\theta^{\mu\nu} \rightarrow 0$  we recover the usual (commutative) geometry and algebra of functions. Points are recovered as any set of states whose spread  $\Delta x^\mu$  in each coordinate  $x^\mu$  goes to zero in the commutative limit. Such a useful set is, e.g., the set of coherent states in each noncommutative (Darboux) pair of coordinates with average values  $x^\mu$ .

Observations like that can form the basis of a complete mapping between noncommutative fields and commutative functions  $f(x)$ , leading to the notion of the ‘symbol’ of  $f(x)$  and the star-product. Specifically, by expressing fields as functions of

the fundamental operators  $x^\mu$  and ordering the various  $x^\mu$  in the expressions for the fields in a prescribed way, using their known commutators, establishes a one-to-one correspondence between functions of operators and ordinary functions.

The ordering that is most usually adopted is the fully symmetric Weyl ordering, in which monomials in the  $x^\mu$  are fully symmetrized. It is simplest to work with the Fourier transforms of functions, since exponentials of linear combinations of  $x^\mu$  are automatically Weyl ordered. So a classical function  $f(x)$ , with Fourier transform  $\tilde{f}(k)$ , is mapped to the operator (noncommutative field)  $f$  as:

$$f = \int dk e^{ik_\mu x^\mu} \quad (9)$$

(the integral over  $k$  is of the appropriate dimensionality). Conversely, the ‘symbol’ (commutative function) corresponding to an operator  $f$  can be expressed as:

$$\tilde{f}(k) = \sqrt{\det(\theta/2\pi)} \operatorname{tr} f e^{-ik_\mu x^\mu} \quad (10)$$

where the above trace is taken over an irreducible representation of the noncommutative coordinates. This reproduces scalar functions. For matrix-valued noncommutative fields  $f$ , acting nontrivially on a direct sum of  $N$  copies of the irreducible representation, the above expression generalizes to

$$\tilde{f}^{ab}(k) = \sqrt{\det(\theta/2\pi)} \sum_n \langle n, a | f e^{-ik_\mu x^\mu} | n, b \rangle \quad (11)$$

where  $|n, a\rangle$  are a complete set of states for the  $a$ -th copy of the irreducible representation, reproducing a matrix function of commutative variables. Hermitian operators  $f$  map to hermitian matrix functions  $f^{ab}(x)$  or, in the case  $N = 1$ , real functions.

One can show that, under the above mapping, derivatives and integrals of noncommutative fields map to the standard commutative ones for their symbol. The product of operators, however, maps to a new function, called the star-product of the corresponding functions [9]:

$$f \leftrightarrow f(x), \quad g \leftrightarrow g(x) \quad \Longrightarrow \quad fg \leftrightarrow (f * g)(x) \quad (12)$$

The star product can be written explicitly in terms of the Fourier transforms of functions as

$$(f * g)(k) = \int dk \tilde{f}(q) \tilde{g}(k - q) e^{\frac{i}{2} \theta^{\mu\nu} k_\mu k_\nu} \quad (13)$$

This is the standard convolution of Fourier transforms, but with an extra phase factor. The resulting  $*$ -product is associative but noncommutative and also nonlocal in the coordinates  $x^\mu$ . The commutator of two noncommutative fields maps to the so-called star, or Moyal, brackets of their symbols.

The above mapping has the advantage that it circumnavigates the conceptual problems of noncommutative geometry by working with familiar objects such as ordinary functions and their integral and derivatives, trading the effects of noncommutativity for a nonlocal, noncommutative function product. It can, however, obscure the beauty and conceptual unification that arises from noncommutativity and make some issues or calculations unwieldy. In what follows, we shall stick with the operator formulation as exposed above. Translation into the  $*$ -product language can always be done at any desired stage.

### 3 Noncommutative gauge theory

Gauge theory on noncommutative spaces becomes particularly attractive [10, 11, 12]. Gauge fields  $A_\mu$  are hermitian operators acting on the representation space. Since they do not depend on  $\partial_i$  they cannot shift the values of  $y^i$ , while they act nontrivially on the fully noncommuting subspace. They have effectively become big matrices acting on the full Fock space with elements depending on the commuting coordinates. Derivatives of these fields are defined through the adjoint action of  $\partial_\mu$

$$\partial_\mu \cdot A_\nu = [\partial_\mu, A_\nu] \quad (14)$$

Using the above formalism, gauge field theory can be built in a way analogous to the commuting case. Gauge transformations are unitary transformations in the full representation space. Restricting  $A_\mu$  to depend on the coordinates only, as above, produces the so-called  $U(1)$  gauge theory.  $U(N)$  gauge theory can be obtained by relaxing this restriction and allowing  $A_\mu$  to also be a function of the  $G^r$  and thus act on the index  $a$ .

#### 3.1 Background-independent formulation

The basic moral of the previous section is that noncommutative gauge theory can be written in a universal way [13, 14, 15]. In the operator formulation no special distinction needs be done between  $U(1)$  and  $U(N)$  theories, nor need gauge and spacetime degrees of freedom be treated distinctly. The fundamental operators of the theory are

$$D_\mu = -i\partial_\mu + A_\mu \quad (15)$$

corresponding to covariant derivatives. Gauge transformations are simply unitary conjugations of the covariant derivative operators by a unitary field  $U$ . That is, the  $D_\mu$  transform covariantly:

$$D_\mu \rightarrow U^{-1} D_\mu U \quad (16)$$

This reproduces the (noncommutative version of the) standard gauge transformation of  $A_\mu$ :

$$A_\mu \rightarrow -iU^{-1}\partial_\mu \cdot U + U^{-1}A_\mu U \quad (17)$$

For the fully noncommutative components, covariant derivative operators assume the form

$$D_\alpha = \omega_{\alpha\beta}x^\beta + A_\alpha = \omega_{\alpha\beta}(x^\beta + \theta^{\beta\gamma}A_\gamma) = \omega_{\alpha\beta}X^\beta \quad (18)$$

The above rewriting is important in various ways. It stresses the fact that, on fully noncommutative spaces, the separation of  $D_\alpha$  into  $x^\alpha$  (coordinate) and  $A_\alpha$  (gauge) is largely arbitrary and artificial: both are operators acting on the Hilbert space on an equal footing, the distinction between ‘derivative’ and ‘coordinate’ having been eliminated. This separation is also gauge dependent, since a unitary transformation will mix the two parts. In effect, gauge transformations mix spatial and gauge degrees of freedom! Further, it is not consistent any more to consider strictly  $SU(N)$  gauge fields. Even if  $A_\mu$  is originally traceless in the  $N$ -dimensional index  $a$ , gauge transformations  $U$  cannot meaningfully be restricted to  $SU(N)$ : the notion of partial trace of an operator with respect to one component of a direct product space makes sense, but the notion of partial determinant does not. A gauge transformation will always generate a  $U(1)$  part for  $A_\mu$ , making  $U(N)$  gauge theory the only theory that arises naturally.

The above rewriting also introduces the ‘covariant coordinate’ field  $X^\alpha$  that combines the ordinary coordinate and gauge fields in a covariant way and is dual to the covariant derivative. Noncommutative gauge theory can be constructed entirely in terms of the  $X^\alpha$ . These, in turn, can be thought of as ‘deformed’ coordinates, the deformation being generated by (the dual of) gauge fields, which alludes to stretching membranes and fluids. All this is relevant in the upcoming story.

Any lagrangian built entirely out of  $D_\mu$  will lead to a gauge invariant action, since the trace will remain invariant under any unitary transformation. The standard Lorentz-Yang-Mills action is built by defining the field strength

$$F_{\mu\nu} = \partial_\mu \cdot A_\nu - \partial_\nu \cdot A_\mu + i[A_\mu, A_\nu] = i[D_\mu, D_\nu] - \omega_{\mu\nu} \quad (19)$$

and writing the standard action

$$S_{LYM} = \frac{1}{4g^2} \text{Tr} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4g^2} \text{Tr} ([D_\mu, D_\nu] + i\omega_{\mu\nu})^2 \quad (20)$$

where Tr also includes integration over commutative components  $y^i$ . In the above we used some c-number metric tensor  $g^{\mu\nu}$  to raise the indices of  $F$ . Note that the operators  $\partial_\alpha$ , understood to act in the adjoint on fields, commute, while the operators  $\partial_\alpha = -i\omega_{\alpha\beta} X^\beta$  have a nonzero commutator equal to

$$[\partial_\alpha, \partial_\beta] = i\omega_{\alpha\beta} \quad (21)$$

This explains the extra  $\omega$ -term appearing in the definition of  $F$  in terms of covariant derivative commutators.

One can, however, just as well work with the action

$$\hat{S}_{LYM} = \frac{1}{4g^2} \text{Tr} \hat{F}_{\mu\nu} \hat{F}^{\mu\nu} = -\frac{1}{4g^2} \text{Tr} [D_\mu, D_\nu] [D^\mu, D^\nu] \quad (22)$$

Indeed,  $\hat{S}$  differs from  $S$  by a term proportional to  $\text{Tr}\omega^2$ , which is an irrelevant (infinite) constant, as well as a term proportional to  $\omega^{\mu\nu} \text{Tr}[D_\mu, D_\nu]$ , which, being the trace of a commutator (a ‘total derivative’), does not contribute to the equations of motion. The two actions lead to the same classical theory. Note that  $\theta^{\mu\nu}$  or  $\omega_{\mu\nu}$  do not appear in the action. These quantities arise only in the commutator of noncommutative coordinates. Since the  $x^\mu$  do not explicitly appear in the action either (being just a gauge-dependent part of  $D_\mu$ ), all reference to the specific noncommutative space has been eliminated! This is the ‘background independent’ formulation of noncommutative gauge theory that stresses its universality.

### 3.2 Superselection of the noncommutative vacuum

How does, then, a particular noncommutative space arise in this theory? The equations of motion for the operators  $D_\mu$  are

$$[D^\mu, [D_\mu, D_\nu]] = 0 \quad (23)$$

The general operator solution of this equation is not fully known. Apart from the trivial solution  $D_\mu = 0$ , it admits as solution all operators with c-number commutators, satisfying

$$[D_\mu, D_\nu] = -i\omega_{\mu\nu} \quad (24)$$

for some  $\omega$ . This is the classical ‘noncommutative vacuum’, where  $D_\mu = -i\partial_\mu$ , and expanding  $D_\mu$  around this vacuum leads to a specific noncommutative gauge theory.

Quantum mechanically,  $\omega_{\mu\nu}$  are superselection parameters and the above vacuum is stable. To see this, assume that the time direction is commutative and consider the collective mode

$$D_\alpha = -i\lambda_{\alpha\beta}\partial_\beta \quad (25)$$

with  $\lambda_{\alpha\beta}$  parameters depending only on time. This mode would change the noncommutative vacuum while leaving the gauge field part of  $D_\alpha$  unexcited.  $\omega$  gets modified into

$$\omega'_{\mu\nu} = \lambda_{\mu\alpha}\omega_{\alpha\beta}\lambda_{\beta\nu} \quad (26)$$

The action implies a quartic potential for this mode, with a strength proportional to  $\text{Tr}1$ , and a kinetic term proportional to  $\text{Tr}\partial_\alpha\partial_\beta$ . (There is also a gauge constraint which does not alter the qualitative dynamical behavior of  $\lambda$ .) Both potential and kinetic terms are infinite, and to regularize them we should truncate each Fock space trace up to some highest state  $\Lambda$ , corresponding to a finite volume regularization (the area of each noncommutative two-dimensional subspace has effectively become  $\Lambda$ ). One can check that the potential term would grow as  $\Lambda^n$  while the kinetic term would grow as  $\Lambda^{n+1}$ . Thus the kinetic term dominates; the above collective degrees of freedom acquire an infinite mass and will remain ‘frozen’ to whatever initial value they are placed, in spite of the nontrivial potential. (This is analogous to the  $\theta$ -angle of the vacuum of four-dimensional nonabelian gauge theories: the vacuum energy depends on  $\theta$  which is still superselected.) Quantum mechanically there is no interference between different values of  $\lambda$  and we can fix them to some c-number value, thus fixing the noncommutativity of space [16]. This phenomenon is similar to symmetry breaking, but with the important difference that the potential is not flat along changes of the ‘broken’ vacuum, and consequently there are no Goldstone bosons.

In conclusion, we can start with the action (22) as the definition of our theory, where  $D_\mu$  are arbitrary operators (matrices) in some space. Gauge theory is then defined as a perturbation around a (stable) classical vacuum. Particular choices of this vacuum will lead to standard noncommutative gauge theory, with  $\theta^{\mu\nu}$  and  $N$  appearing as vacuum parameters. Living in any specific space and gauge group amounts to landscaping!

### 3.3 Noncommutative Chern-Simons action

A particularly useful and important type of action in gauge theory is the Chern-Simons term [17]. This is a topological action, best written in terms of differential forms. In the commutative case, we define the one- and two-forms

$$A = iA_\mu dx^\mu, \quad F = dA + A^2 = \frac{i}{2} \left( \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu] \right) dx^\mu dx^\nu \quad (27)$$

The Chern-Simons action  $S_{2n+1}$  is the integral of the  $2n+1$ -form  $C_{2n+1}$  satisfying

$$dC_{2n+1} = \text{tr}F^{n+1} \quad (28)$$

By virtue of (28) and the gauge invariance of  $\text{tr}F^n$  it follows that  $S_{2n+1}$  is gauge invariant up to total derivatives, since, if  $\delta$  stands for an infinitesimal gauge transformation,

$$\delta dC_{2n+1} = \delta dC_{2n+1} = \delta \text{tr}F^n = 0, \quad \text{so} \quad \delta C_{2n+1} = d\Omega_{2n} \quad (29)$$



The integrated action is therefore invariant under infinitesimal gauge transformations. Large gauge transformations may lead to an additive change in the action and they usually imply a quantization of its coefficient [17, 18]. As a result, the equations of motion derived from this action are gauge covariant and read

$$\frac{\delta S_{2n+1}}{\delta A} = \frac{\delta}{\delta A} \int C_{2n+1} = (n+1)F^n \quad (30)$$

The above can be considered as the defining relation for  $C_{2n+1}$ .

We can define corresponding noncommutative Chern-Simons actions [19]-[27]. To this end, we shall adopt the differential form language [14] and define the usual basis of one-forms  $dx^\mu$  as a set of formal anticommuting parameters with the property

$$dx^\mu dx^\nu = -dx^\nu dx^\mu, \quad dx^{\mu_1} \dots dx^{\mu_d} = \epsilon^{\mu_1 \dots \mu_d} \quad (31)$$

Topological actions do not involve the metric tensor and can be written as integrals of  $d$ -forms. The only dynamical objects available in noncommutative gauge theory are  $D_\mu$  and thus the only form that we can write is

$$D = idx^\mu D_\mu = d + A \quad (32)$$

where we defined the exterior derivative and gauge field one-forms

$$d = dx^\mu \partial_\mu, \quad A = idx^\mu A_\mu \quad (33)$$

(note that both  $D$  and  $A$  as defined above are antihermitian). The action of the exterior derivative  $d$  on an operator  $p$ -form  $H$ ,  $d \cdot H$ , yields the  $p+1$ -form  $dx^\mu [\partial_\mu, H]$  and is given by

$$d \cdot H = dH - (-)^p Hd \quad (34)$$

In particular, on the gauge field one-form  $A$  it acts as

$$d \cdot A = dA + Ad \quad (35)$$

Correspondingly, the covariant exterior derivative of  $H$  is

$$D \cdot H = DH - (-)^p HD \quad (36)$$

As a result of the noncommutativity of the operators  $\partial_\mu$ , the exterior derivative operator is not nilpotent but rather satisfies

$$d^2 = \omega, \quad \omega = \frac{i}{2} dx^\mu dx^\nu \omega_{\mu\nu} \quad (37)$$

We stress, however, that  $d \cdot$  is still nilpotent since  $\omega$  commutes with all operator forms:

$$d \cdot d \cdot H = [d, [d, H]_{\mp}]_{\pm} = \pm[\omega, H] = 0 \quad (38)$$

The two-form  $\hat{F} = \frac{i}{2} dx^\mu dx^\nu \hat{F}_{\mu\nu}$  is simply

$$\hat{F} = D^2 = \frac{1}{2} D \cdot D = \omega + dA + Ad + A^2 = \omega + F \quad (39)$$

where  $F = \frac{i}{2} dx^\mu dx^\nu F_{\mu\nu}$  is the conventionally defined field strength two-form.

The most general  $d$ -form that we can write involves arbitrary combinations of  $D$  and  $\omega$ . If, however, we adopt the view that  $\omega$  should arise as a superselection (vacuum) parameter and not as a term in the action, the unique form that we can write is  $D^d$  and the unique action

$$\hat{S}_d = \frac{d+1}{2d} \text{Tr} D^d = \text{Tr} C_d \quad (40)$$

This is the Chern-Simons action. The coefficient was chosen to conform with the commutative definition, as will be discussed shortly. In even dimensions  $\hat{S}_d$  reduces to the trace of a commutator  $\text{Tr}[D, D^{d-1}]$ , a total derivative that does not affect the equations of motion and corresponds to a topological term. In odd dimensions it becomes a nontrivial action.

$\hat{S}_d$  is by construction gauge invariant. To see that it also satisfies the defining property of a Chern-Simons form (30) is almost immediate:  $\delta/\delta A = \delta/\delta D$  and thus, for  $d = 2n + 1$ :

$$\frac{\delta}{\delta A} \text{Tr} D^{2n+1} = (2n+1) D^{2n} = (2n+1) \hat{F}^n \quad (41)$$

So, with the chosen normalization in (40) we have the defining condition (30) with  $\hat{F}$  in the place of  $F$ . What is less obvious is that  $\hat{S}_D$  can be written entirely in terms of  $F$  and  $A$  and that, for commutative spaces, it reduces to the standard Chern-Simons action. To achieve that, one must expand  $C_D$  in terms of  $d$  and  $A$ , make use of the cyclicity of trace and the condition  $d^2 = \omega$  and reduce the expressions into ones containing  $dA + Ad$  rather than isolated  $d$  s. The condition

$$\text{Tr} \omega^n d = 0 \quad (42)$$

which is a result of the fact that  $\partial_\mu$  is off-diagonal for both commuting and noncommuting dimensions, can also be used to get rid of overall constants. This is a rather involved procedure for which we have no algorithmic approach. (Specific cases will be worked out later.) Note, further, that the use of the cyclicity of trace implies that we dismiss total derivative terms (traces of commutators). Such terms do not affect the equations of motion. For  $d = 1$  the result is simply

$$\hat{S}_1 = \text{Tr} A \quad (43)$$

which is the ‘abelian’ one-dimensional Chern-Simons term. For  $d = 3$  we obtain

$$\hat{S}_3 = \text{Tr} (AF - \frac{1}{3} A^3) + 2\text{Tr}(\omega A) \quad (44)$$

where we used the fact that  $\text{Tr}[A(dA + Ad)] = 2\text{Tr}(A^2 d)$ . The first term is the noncommutative version of the standard three-dimensional Chern-Simons term, while the second is a lower-dimensional Chern-Simons term involving explicitly  $\omega$ .

We can get the general expression for  $\hat{S}_d$  by referring to the defining relation. This reads

$$\frac{\delta}{\delta A} \hat{S}_{2n+1} = (n+1) \hat{F}^n = (n+1)(F + \omega)^n = (n+1) \sum_{k=0}^n \binom{n}{k} \omega^{n-k} F^k \quad (45)$$

and by expressing  $F^k$  as the  $A$ -derivative of the standard Chern-Simons action  $S_{2k+1}$  we get

$$\frac{\delta}{\delta A} \left\{ \hat{S}_{2n+1} - \sum_{k=0}^n \binom{n+1}{k+1} \omega^{n-k} S_{2k+1} \right\} = 0 \quad (46)$$

So the expression in brackets must be a constant, easily seen to be zero by setting  $A = 0$ . We therefore have

$$\hat{S}_{2n+1} = \sum_{k=0}^n \binom{n+1}{k+1} \text{Tr} \omega^{n-k} C_{2k+1} \tag{47}$$

We observe that we get the  $2n + 1$ -dimensional Chern-Simons action plus all lower-dimensional actions with tensors  $\omega$  inserted to complete the dimensions. Each term is separately gauge invariant and we could have chosen to omit them, or include them with different coefficient. It is the specific combination above, however, that has the property that it can be reformulated in a way that does not involve  $\omega$  explicitly. The standard Chern-Simons action can also be written in terms of  $D$  alone by inverting (47):

$$S_{2n+1} = (n + 1) \text{Tr} \int_0^1 D(t^2 D^2 - \omega)^n dt = \text{Tr} \sum_{k=0}^n \binom{n+1}{k+1} \frac{k+1}{2k+1} (-\omega)^{n-k} D^{2k+1} \tag{48}$$

For example, the simplest nontrivial noncommutative action in 2+1 dimensions reads

$$S_3 = \text{Tr} \left( \frac{2}{3} D^3 - 2\omega D \right) \tag{49}$$

The above can be written more explicitly in terms of the two spatial covariant derivatives  $D_{1,2}$ , which are operators acting on the noncommutative space, and the temporal covariant derivative  $D_0 = dt(\partial_t + iA_0)$ , which contains a proper derivative operator in the commutative direction  $x^0 = t$  and a noncommutative gauge field  $A_0$ :

$$S_3 = \int dt 2\pi\theta \text{Tr} \left\{ \epsilon^{ij} (\dot{D}_i + i[A_0, D_i]) D_j + \frac{2}{\theta} A_0 \right\} \tag{50}$$

Note that the overall coefficient of the last, linear term is independent of  $\theta$ .

We also point out a peculiar property of the Chern-Simons form  $\hat{C}_{2n+1}$ . Its covariant derivative yields  $\hat{F}^{n+1}$ :

$$D \cdot \hat{C}_{2n+1} = D\hat{C}_{2n+1} + \hat{C}_{2n+1}D = \frac{2n+2}{2n+1} \hat{F}^{n+1} \tag{51}$$

A similar relation holds between  $C_d$  (understood as the form appearing inside the trace in the right hand side of (48)) and  $F$ . Clearly the standard Chern-Simons form does not share this property. Our  $C_d$  differs from the standard one by commutators that cannot all be written as ordinary derivatives (such as, *e.g.*,  $[d, dA]$ ). These unconventional terms turn  $C_d$  into a covariant quantity that satisfies (51).

### 3.4 Level quantization for the noncommutative Chern-Simons action

We conclude our consideration of the noncommutative Chern-Simons action by considering the quantization requirements for its coefficient [26, 27].

In the commutative case, a quantization condition for the coefficient of non-abelian Chern-Simons actions ('level quantization') is required for global gauge invariance. This has its roots in the topology of the group of gauge transformations in the given manifold. *E.g.*, for the 3-dimensional term, the fact that  $\pi_3[S(N)] = Z$  for any  $N > 1$  implies the existence of topologically nontrivial gauge transformations and corresponding level quantization.

For the noncommutative actions we have not studied the topology of the gauge group. This would appear to be a hard question for a ‘fuzzy’ noncommutative space, but in fact it is well-defined and easy to answer: gauge transformations are simply unitary transformations on the full representation space on which  $X^\mu$  or  $D_\mu$  act. This space is infinite dimensional, so we are dealing with (some version of)  $U(\infty)$ . Two observations, however, elucidate the answer. First, for odd-dimensional noncommutative spaces there is always one (and in general only one) commutative dimension  $t$ , conventionally called time; and second, if we require gauge transformations to act trivially at infinity, we are essentially restricting the corresponding unitary operators to have finite support on the representation space and be bounded. So the relevant gauge transformations are essentially  $U(N)$  matrices of the form  $U(t)$ . The relevant topology is  $S^1 \rightarrow U(N)$  and is nontrivial due to the  $U(1)$  factor in  $U(N)$ :

$$\pi_1[U(N)] = \pi_1[U(1)] = Z \quad (52)$$

This is true for *any* noncommutative gauge theory, abelian or nonabelian. A ‘winding number one’ transformation would be a matrix of the form

$$U(t) = e^{i\frac{2\pi}{N}t}\tilde{U}(t), \quad t \in [0, 1] \quad (53)$$

with  $\tilde{U}$  an  $SU(N)$  matrix satisfying  $\tilde{U}(0) = 1$  and  $\tilde{U}(1) = \exp(-i\frac{2\pi}{N})$ , a  $Z_N$  matrix. This satisfies  $U(0) = U(1) = 1$  but cannot be smoothly deformed to  $U(t) = 1$ .

What is the change, if any, of the noncommutative Chern-Simons action under the above transformation? We may look at the explicit form (50) of  $S_3$  to decide it. The first, cubic term is completely gauge invariant. Indeed, under a gauge transformation the quantity inside the trace and integral transforms covariantly

$$\epsilon^{ij}(\dot{D}_i + i[A_0, D_i])D_j \rightarrow U(t)^{-1} \left[ \epsilon^{ij}(\dot{D}_i + i[A_0, D_i])D_j \right] U(t) \quad (54)$$

and upon tracing it remains invariant. The term  $A_0$ , however, transforms as

$$A_0 \rightarrow U(t)^{-1}A_0U(t) - iU(t)^{-1}\dot{U}(t) \quad (55)$$

The last term gives a nontrivial contribution to the action equal to

$$\Delta S_3 = -i4\pi \int_0^1 dt \operatorname{tr} U(t)^{-1}\dot{U}(t) \quad (56)$$

The  $SU(N)$  part  $\tilde{U}$  of  $U(t)$  does not contribute in the above, since  $\tilde{U}^{-1}\dot{\tilde{U}}$  is traceless. The  $U(1)$  factor, however, contributes a part equal to

$$\Delta S_3 = -i4\pi \int_0^1 dt i\frac{2\pi}{N}\operatorname{tr} 1 = 8\pi^2 \quad (57)$$

The coefficient of the action  $\lambda$  should be such that the overall change should be quantum mechanically invisible, that is, a multiple of  $2\pi$ . We get

$$\lambda 8\pi^2 = 2\pi n \quad \text{or} \quad \lambda = \frac{n}{4\pi} \quad (58)$$

with  $n$  an integer.

The above quantization condition is independent of  $\theta$  and conforms with the level quantization of the commutative nonabelian Chern-Simons theory. It also holds

for the *abelian* (or, rather,  $U(1)$ ) theory, for which there is no quantization in the commutative case. In the commutative limit the corresponding topologically nontrivial gauge transformations become singular and decouple from the theory, thus eliminating the need for quantization. This result will be relevant in the upcoming considerations of the quantum Hall effect.

## 4 Connection with fluid mechanics

At this point we take a break from noncommutative gauge theory to bring into the picture fluid mechanics and review its two main formulations, Euler and Lagrange. As will become apparent, the two subjects are intimately related. Already we saw that noncommutative gauge theory can be formulated in terms of covariant deformed coordinate operators  $X^\mu$ . These parallel the spatial coordinates of particle fluids, with the undeformed background coordinates  $x^\mu$  playing the role of body-fixed labels of the particles.

### 4.1 Lagrange and Euler descriptions of fluids

We start with a summary review of the two main formulations of fluid mechanics, the particle-fixed (Lagrange) and space-fixed (Euler) descriptions. For more extensive reviews see [28, 29].

A fluid can be viewed as a dense collection of (identical) particles moving in some  $d$ -dimensional space, evolving in time  $t$ . The Lagrange description uses the coordinates of the particles comprising the fluid:  $X^i(x, t)$ . These are labeled by a set of parameters  $x^i$ , which are the coordinates of some fiducial reference configuration and are called particle-fixed or comoving coordinates. They serve, effectively, as particle ‘labels’. Summation over particles amounts to integration over the comoving coordinates  $x$  times the density of particles in the fiducial configuration  $\rho_0(x)$ , which is usually taken to be homogeneous.

In the Euler description the fluid is described by the space-time-dependent density  $\rho(r, t)$  and velocity fields  $v^i(r, t)$  at each point of space with coordinates  $r^i$ . The two formulations are related by considering the particles at space coordinates  $r^i$ , that is,  $X^i = r^i$ , and expressing the density and velocity field in terms of the Lagrange variables. We assume sufficient regularity so that (single-valued) inverse functions  $\chi^i(r, t)$  exist:

$$X^i(t, x) \Big|_{x=\chi(t, r)} = r^i \quad (59)$$

$X^i(x, t)$  provides a mapping of the fiducial particle position  $x^i$  to position at time  $t$ , while  $\chi^i(r, t)$  is the inverse mapping. The Euler density then is defined by

$$\rho(r, t) = \rho_0 \int dx \delta(X(x, t) - r) . \quad (60)$$

(The integral and the  $\delta$ -function carry the dimensionality of the relevant space.) This evaluates as

$$\frac{1}{\rho(r, t)} = \frac{1}{\rho_0} \det \frac{\partial X^i(x, t)}{\partial x^j} \Big|_{x=\chi(r, t)} \quad (61)$$

which is simply the change of volume element from fiducial to real space. The Euler velocity is

$$v^i(r, t) = \dot{X}^i(x, t) \Big|_{x=\chi(r, t)} \quad (62)$$

where overdot denotes differentiation with respect to the explicit time dependence. (Evaluating an expression at  $x = \chi(r, t)$  is equivalent to eliminating  $x$  in favor of  $X$ , which is then renamed  $r$ .)

The number of particles in the fluid is conserved. This is a trivial (kinematical) condition in the Lagrange formulation, where comoving coordinates directly relate to particles. In the Euler formulations this manifests through conservation of the particle current  $j^i = \rho v^i$ , given in terms of Lagrange variables by

$$v^i(r, t) = \rho_0 \int dx \dot{X}^i(r, t) \delta(X(x, t) - r) \quad (63)$$

As a consequence of the above definition it obeys the continuity equation

$$\dot{\rho} + \partial_i j^i = 0. \quad (64)$$

The kinetic part of the lagrangian  $K$  for the Lagrange variables is simply the single-particle lagrangian for each particle in terms of the particle coordinates,  $K_{sp}(X)$ , summed over all particles.

$$K = \rho_0 \int dx K_{sp}(X(x, t)). \quad (65)$$

The exact form of  $K_{sp}$  depends on whether the particles are relativistic or non-relativistic, the presence of magnetic fields etc. As an example, the kinetic term for a non-relativistic plasma in an external magnetic field generated by an electromagnetic vector potential  $\mathcal{A}_i$  is

$$K = \rho_0 \int dx \left[ \frac{1}{2} m g_{ij}(X) \dot{X}^i \dot{X}^j + q \mathcal{A}_i(X, t) \dot{X}^i \right] \quad (66)$$

with  $m$  and  $q$  the mass and charge of each fluid particle and  $g_{ij}$  the metric of space.

Single-particle (external) potentials can be written in a similar way, while many-body and near-neighbor (density dependent) potentials will be more involved.

## 4.2 Reparametrization symmetry and its noncommutative avatar

The Lagrange description has an obvious underlying symmetry. Comoving coordinates are essentially arbitrary particle labels. All fluid quantities are invariant under particle relabeling, that is, under reparametrizations of the variables  $x^i$ , provided that the density of the fiducial configuration  $\rho_0$  remains invariant. Such transformations are volume-preserving diffeomorphisms of the variables  $x^i$ .

For the minimal nontrivial case of two spatial dimensions, this symmetry corresponds to area-preserving diffeomorphisms. They can be thought of as canonical transformations on a two-dimensional phase space and are parametrized by a function of the two spatial variables, the generator of canonical transformation. Infinitesimal transformations are written

$$\delta x^i = \epsilon^{ij} \frac{\partial f}{\partial x^j} \quad (67)$$

with  $f(x)$  the generating function. Obviously  $\delta x^i$  satisfies the area-preserving condition

$$\det \frac{\partial(x^i + \delta x^i)}{\partial x^j} = 1 \quad \text{or} \quad \frac{\partial \delta x^i}{\partial x^i} = 0 \quad (68)$$

The same condition can be written in an even more suggestive way. Define a canonical structure for the two-dimensional space in terms of the Poisson brackets

$$\{x^1, x^2\} = \theta \quad \text{or} \quad \{x^i, x^j\} = \theta \epsilon^{ij} = \theta^{ij} \quad (69)$$

for some constant  $\theta$ . Rescaling  $f$  by a factor  $\theta^{-1}$ , we can re-write  $\delta x^i$  as

$$\delta x^i = \theta^{ij} \partial_j f = \{x^i, f\} \quad (70)$$

Similarly, the transformation of the fundamental (Lagrange) fluid variables under the above redefinition is

$$\delta X^i = \partial_j X^i \delta x^j = \theta^{jk} \partial_j X^i \partial_k f = \{X^i, f\} \quad (71)$$

The above look like the classical analog (or precursor) of the gauge transformations of the covariant noncommutative gauge coordinates  $X^i$  of the previous sections. This is not accidental: the area-preserving transformations for the fluid correspond to relabeling the parameters  $x$  and do not generate a physically distinct fluid configurations. They represent simply a redundancy in the description of the fluid in terms of Lagrange coordinates; that is, a gauge symmetry. Physical fluid quantities, such as the Euler variables, or the fluid lagrangian, are expressed as integrals of quantities transforming ‘covariantly’ under the above transformation; that is, transforming by the Poisson bracket of the quantity with the generator of the transformation  $f$ , as in (71). They are, therefore, invariant under such transformations; that is, gauge invariant.

The analogy with noncommutative gauge theory becomes manifest by writing the Lagrange particle coordinates in terms of their deviation from the fiducial coordinates [30]-[33]:

$$X^i(x, t) = x^i + a^i(x, t) = x^i + \theta^{ij} A_j(x, t) \quad (72)$$

The deviation  $a^i$ , and its dual  $A_i$  do not transform covariantly any more; rather

$$\delta A_i = \partial_i f + \{A_i, f\} \quad (73)$$

The similarity with the gauge transformation of a gauge field is obvious. The duals of the  $X^i$

$$D_i = \omega_{ij} X^j = \omega_{ij} x^j + A_i \quad (74)$$

obviously correspond to covariant derivatives (although at this stage they are just rewritings of the comoving particle coordinates). The analog of the field strength is

$$\hat{F}_{ij} = \{D_i, D_j\} = \omega_{ij} + \partial_i A_j - \partial_j A_i + \{A_i, A_j\} \quad (75)$$

This is related to the fluid density, which in the Poisson bracket formulation reads

$$\frac{\rho_0}{\rho} = \det \frac{\partial X^k(x, t)}{\partial x^l} = \frac{1}{\theta} \{X^1, X^2\} \quad (76)$$

The field strength calculates as:

$$\hat{F}_{ij} = \omega^{ij} \{X^1, X^2\} = \frac{\rho_0}{\rho} \epsilon_{ij} \quad (77)$$

The field strength essentially becomes the (inverse) fluid density!

Similar considerations generalize to higher dimensions, with one twist: canonical transformations, the classical version of noncommutative gauge transformations, are only a symplectic subgroup of full volume-preserving diffeomorphisms. Higher-dimensional noncommutative gauge theory is analogous to a special version of fluid mechanics that enjoys a somewhat limited particle relabeling invariance. For the purposes of describing the quantum Hall effect, an essentially two-dimensional situation, this is inconsequential.

### 4.3 Gauging the symmetry

In the above discussion the role of time was not considered. The particle relabeling ( $x$ -space reparametrization) considered above were time-independent. Time-dependent transformations are not, a priori, invariances of the fluid since they introduce extra, nonphysical terms in the particle velocities  $\dot{X}^i(x, t)$ . To promote this transformation into a full space-time gauge symmetry we must gauge time derivatives by introducing a temporal gauge field  $A_0$ :

$$D_0 X^i = \dot{x}^i + \{A_0, X^i\} \quad (78)$$

Under the transformation (71) with a time-dependent function  $f$  the above derivative will transform covariantly

$$\delta D_0 X^i = \{D_0 X^i, f\} \quad (79)$$

provided that the gauge field  $A_0$  transforms as

$$\delta A_0 = \dot{f} + \{A_0, f\} \quad (80)$$

This gauging, however, has dynamical consequences. We can gauge fix the theory by choosing the temporal gauge, putting  $A_0 = 0$ . The action becomes identical to the ungauged action, with the exception that now we have to satisfy the Gauss law for the gauge-fixed symmetry, that is, the equation of motion for the reduced field  $A_0$ . The exact form of the constraint depends on the kinetic term of the lagrangian for the fluid:

$$G = \{\dot{X}^i, \frac{\partial K}{\partial \dot{X}^i}\} = 0 \quad (81)$$

As an example, for the plasma of (4.1) the Gauss law reads

$$G = \{\dot{X}^i, m g_{ij}(X) \dot{X}^j + q \mathcal{A}_i(X)\} = 0 \quad (82)$$

Interesting two-dimensional special cases are ( $g_{ij} = \delta_{ij}$ ,  $q = 0$ ), when

$$G = \{\dot{X}^i, X^i\} \quad (83)$$

and the ‘lowest Landau level’ case of massless particles in a constant magnetic field ( $m = 0$ ,  $\mathcal{A}_i = (B/2)\epsilon_{ij}X^j$ ), when

$$G = \{X^1, X^2\} = 0 \quad (84)$$

We conclude by mentioning that the fluid structure we described in this section can also be interpreted as membrane dynamics. Indeed, a membrane is, in principle, a sheet of fluid in a higher-dimensional space. A two-dimensional membrane in two space dimensions is space-filling, and thus indistinguishable from a fluid, the density expressing the way in which the membrane shrinks or expand locally. The full correspondence of membranes, noncommutative (matrix) theory and fluids, relativistic and non-relativistic, has been examined elsewhere. We shall not expand on it here.



#### 4.4 Noncommutative fluids and the Seiberg-Witten map

In the previous section we alluded to the connection between noncommutative gauge theory and fluid mechanics. It is time to make the connection explicit [33]. We shall work specifically in two (flat) spatial dimensions, as the most straightforward case and relevant to the quantum Hall effect.

The transition from (classical) fluids to noncommutative fluids is achieved the same way as the transition from classical to quantum mechanics. We promote the canonical Poisson brackets introduced in the previous section to (operator) commutators. All Poisson brackets that appear become commutators:

$$\{ , \} \rightarrow -i[ , ] \quad (85)$$

So the comoving parameters satisfy

$$[x^i, x^j] = i\theta^{ij} \quad (86)$$

They have become a noncommutative plane. This means that the particle labels cannot have ‘sharp’ values and the pinpointing of particles in the fluid is no more possible. In effect, we have a ‘fuzzification’ of the underlying fluid particles and a corresponding ‘fuzzy’ fluid.

The remaining structure smoothly goes over to noncommutative gauge theory, as already alluded. We assume that the noncommutative coordinates  $x^1, x^2$  act on a single irreducible representation of their Heisenberg algebra; this effectively assigns a single particle state for ‘point’ of space (each state in the representation). Inclusion of multiple copies of the irreducible representations would correspond to multiple particle states per ‘point’ of space and would endow the particles with internal degrees of freedom.

Integration over the comoving parameters becomes  $2\pi\theta$  times trace over the representation space. Summation over particles, then, becomes

$$\sum_{\text{particles}} = \rho_0 \int dx \rightarrow 2\pi\theta\rho_0 \text{Tr} \quad (87)$$

The parameter  $\theta$ , or its inverse  $\omega$ , was introduced arbitrarily and plays no role in the fluid description. This is similar to the background-independent formulation of noncommutative gauge theory in terms of covariant derivatives or coordinates. Presently, we relate  $\theta$  to the inverse density of the fiducial configuration  $\rho_0^{-1}$

$$2\pi\theta = \frac{1}{\rho_0} \quad (88)$$

in which case the factor in the preceding equation disappears. Particle summation becomes a simple trace, so particles are identified with states in the representation space. This relation between fiducial density and noncommutativity parameter will always be assumed to hold from now on.

The Lagrange coordinates of particles  $X^i$  and the gauge field  $A_0$  are functions of the underlying ‘fuzzy’ (noncommutative) particle labels, and thus become noncommutative fields. Area-preserving reparametrizations, which are canonical transformations in the classical case, become unitary transformations in the noncommutative case (think, again, of quantum mechanics). Operators  $X^i$  transform by unitary conjugations; infinitesimally,

$$\delta X^i = i[f, X^i] \quad (89)$$

The deviations of  $X^i$  from the fiducial coordinates  $x^i$ , on the other hand, as defined in (72), and the temporal gauge field pick up extra terms and transform as proper gauge fields:

$$\delta A_\mu = \partial_\mu f - i[A_\mu, f] \quad (90)$$

The remaining question is the form of the (gauge invariant) lagrangian that corresponds to the noncommutative fluid. This depends on the specific fluid dynamics and will be dealt with in the next section. Before we go there, we would like to examine further the properties of the noncommutative fluid that derives from the present construction. Just because the underlying particles become fuzzy does not necessarily mean that the emerging fluid cannot be described in traditional terms. Indeed, fluids are dense distributions of particles and we are not supposed to be able to distinguish individual particles in any case. The Euler description, which talks about collective fluid properties like density and velocity, remains valid in the noncommutative case as we shall see.

The noncommutative version of equation (76) for the density becomes (with  $2\pi\theta\rho_0 = 1$ )

$$[X^1, X^2] = \frac{i}{2\pi\rho} \quad (91)$$

This relation would suggest that the density, too, becomes a noncommutative field. The difficulty with this expression is that it gives the density as a function of the underlying comoving coordinates, which we know are noncommutative. A better expression is (60), which gives the density directly at a point of space  $r$ . This formula directly transcribes into

$$\rho(r, t) = \text{Tr}\delta(X - r) \quad (92)$$

in the noncommutative case.  $r$  is still an ordinary space variable, and the trace eliminates the operator nature of the expression in the right hand side, rendering a classical function of  $r$  and  $t$ . The only difficulty is in the definition of the delta function for the noncommutative argument  $X^i - r^i$ : the various  $X^i$  (two in our case) are operators and do not commute, so there are ordering issues in defining any function. In fact, the operator  $\delta(X - r)$  may not even be hermitian unless properly ordered, which would produce a complex density.

In dealing with such problems, a procedure similar to the definition of the ‘symbol’ of a noncommutative field is followed: a standard ordering of all monomials involving various  $X^i$ s is prescribed. The Weyl (totally symmetrized) ordering is usually adopted. Under this ordering, the delta function above is defined as

$$\delta(X - r) = \int dk e^{ik_i(r^i - X^i)} \quad (93)$$

where  $k_i$  are classical (c-number) Fourier integration parameters. The above operator has also the advantage of being hermitian. The spatial Fourier transform of the density with respect to  $r$  is simply

$$\rho(k, t) = \text{Tr} e^{-ik_i X^i} \quad (94)$$

In a similar vein, we use the classical expression for the particle current

$$j^i(r, t) = \rho_0 \int dx \dot{X}^i \delta(X - r) \quad (95)$$

to write the corresponding expression for the noncommutative fluid as

$$j^i(k, t) = \text{Tr} D_0 X^i e^{-ik_i X^i} \quad (96)$$

In the above, we used the covariant time derivative in order to make the expression explicitly gauge invariant. The corresponding current is real, as the trace ensures that the change of ordering between  $D_0 X$  and the exponential is immaterial.

The crucial observation is that the above density and current still satisfy the continuity equation, which in Fourier space becomes

$$\dot{\rho} + ik_i j^i = 0 \quad (97)$$

The proof is straightforward and relies on the following two facts, true due to the cyclicity of trace:

$$\frac{d}{dt} \text{Tr} e^{-ik_i X^i} = -i \text{Tr} k_j \dot{X}^j e^{-ik_i X^i} \quad (98)$$

and

$$\text{Tr} [A_0, k_j X^j] e^{-ik_i X^i} = 0 \quad (99)$$

The noncommutative fluid, therefore, has an Euler description in terms of a traditional conserved particle density and current.

The above observation is the basis for a mapping between commutative and noncommutative gauge theories, which first arose in the context of string theory and is known as the Seiberg-Witten map [5]. The key observation is that, in 2+1 dimensions, a conserved current can be written in terms of its dual two-form, which then satisfies the Bianchi identity. Specifically, define

$$J_{\mu\nu} = \epsilon_{\mu\nu\lambda} j^\lambda \quad (100)$$

where  $j^0 = \rho$ . Then, due to the continuity equation  $\partial_\mu j^\mu = 0$ ,  $J_{\mu\nu}$  satisfies

$$\partial_\mu J_{\nu\lambda} + \text{cyclic perms.} = 0 \quad \text{or} \quad dJ = 0 \quad (101)$$

This means that  $J$  can be considered as an abelian field strength, which allows us to define an abelian commutative gauge field  $\tilde{A}_\mu$ . The reference configuration of the fluid, in which particles are in their fiducial positions  $X^i = x^i$  and corresponds to vanishing noncommutative gauge field, gives  $j_0^\mu = (\rho_0, 0, 0)$  or  $J_0 = \rho_0 dx^1 dx^2$ . If we want to have this configuration correspond to vanishing abelian gauge field  $\tilde{F}_{\mu\nu}$ , we have to define

$$\tilde{F} = J - J_0 \quad (102)$$

or, more explicitly

$$\tilde{F}_{0i} = \epsilon_{ik} j^k, \quad \tilde{F}_{ij} = \epsilon_{ij}(\rho - \rho_0) \quad (103)$$

Substituting the explicit expressions (94,96) for  $\rho$  and  $j^i$ , and expressing  $X^i$  in them in terms of noncommutative fields, gives an explicit mapping between the noncommutative fields  $A_\mu$  and the commutative fields  $\tilde{A}_\mu$ .

Similar considerations extend to higher dimensions but, again, we shall not dwell on them here [35, 34, 35]. The moral lesson of the above is that the Lagrange formulation of fuzzy fluids is inherently noncommutative, while the Euler formulation is commutative. The Seiberg-Witten map between them becomes the transition from the particle-fixed Lagrange to the space-fixed Euler formulation.

## 5 The noncommutative description of quantum Hall states

We reach, now, one of the main topics of this presentation. Is the above useful to anything? Can we use it to describe or solve any interesting physical system or does it remain an interesting peculiarity?

To find an appropriate application, we must look for systems with ‘fuzzy’ particles. This is not hard: quantum mechanical particles on their phase spaces are fuzzy, due to Heisenberg uncertainty. This can be carried through, and eventually leads to the description of one-dimensional fermions in terms of matrix models.

A more interesting situation arises in lowest Landau level physics, in which particles become fuzzy on the *coordinate* space. Spatial coordinates become noncommuting when restricted to the lowest Landau level [36, 17], already introducing a noncommutative element (although quite distinct from the one introduced in the sequel). This is also the setting for the description of quantum Hall states and will be the topic of the present section.

### 5.1 Noncommutative Chern-Simons description of the quantum Hall fluid

The system to be described consists of a large number  $N \rightarrow \infty$  of electrons (we take their charge  $e = 1$ ) on the plane in the lowest Landau level of an external constant magnetic field  $B$ . Upon proper dynamical conditions, they form quantum Hall states (for a review of the quantum Hall effect see [38].) According to the observations of the previous section, we can parametrize their coordinates as a fuzzy fluid in terms of two noncommutative Lagrange coordinates (infinite hermitian ‘matrices’)  $X^i$ ,  $i = 1, 2$ , that is, by two operators on an infinite Hilbert space. The density of these electrons is not fixed at this point, but will eventually relate to the noncommutativity parameter as  $\rho_0 = 1/2\pi\theta$ .

The action is the noncommutative fluid analog of the gauge action of massless particles in an external constant magnetic field. In the symmetric gauge for the magnetic field, this would read

$$S = \int dt \frac{B}{2} \text{Tr} \{ \epsilon_{ij} (D_0 X^i X^j) \} = \int dt \frac{B}{2} \text{Tr} \{ \epsilon_{ij} (\dot{X}^i + i[A_0, X^i]) X^j + \} \quad (104)$$

The above expression was made gauge invariant by gauging the time derivative and introducing a noncommutative temporal gauge field  $A_0$ . As explained in previous sections, however, this introduces a Gauss law constraint, which in the present case reads

$$[X^1, X^2] = 0 \quad (105)$$

This is undesirable in many ways. The would-be noncommutative coordinates become commutative, eliminating the fuzziness of the description. More seriously, the density of the fluid classically becomes singular, as can be seen from the expression (91) for the inverse fluid density. (It can also be deduced from the commutative expression (94), although in a slightly more convoluted way.)

Taking care of the above difficulty also gives the opportunity to introduce an important piece of physics for the system: fractional quantum Hall states (Laughlin states, in their simplest form) are incompressible and have a constant spatial density  $\rho_0$ . The *filling fraction*  $\nu$  of the state is defined as the fraction of the Landau level density  $\rho_{LL} = B/2\pi$  that  $\rho_0$  represents:

$$\nu = \frac{\rho_0}{\rho_{LL}} = \frac{2\pi\rho}{B} = \frac{1}{\theta B} \quad (106)$$

where the noncommutative parameter  $\theta$  is related to the desired fluid density in the standard way, spelled out again as

$$\rho_0 = \frac{1}{2\pi\theta} \quad (107)$$

We can introduce this constant density  $\rho_0$  in the system by modifying the Gauss law constraint by an appropriate constant, achieved by adding a term linear in  $A_0$ . The resulting action reads

$$S = \int dt \frac{B}{2} \text{Tr} \left\{ \epsilon_{ij} (\dot{X}^i + i[A_0, X^i]) X^j + 2\theta A_0 \right\} \quad (108)$$

This was first proposed by Susskind [32], motivated by the earlier, classical mapping of the quantum Hall fluid to a gauge action [30] and related string theory work [39]. The equation of motion for  $A_0$ , now, imposes the Gauss law constraint

$$[X^1, X^2] = i\theta \quad (109)$$

essentially identifying  $X^1, X^2$  with a noncommutative plane.

Interestingly, the above action is exactly the noncommutative CS action in 2+1 dimensions! A simple comparison of expression (50) and (108) above reveals that they are the same, upon identifying  $\theta D_i = \epsilon_{ij} X^j$ . The coefficient of the CS term  $\lambda$  relates to  $B$  and the filling fraction as

$$\lambda = \frac{B\theta}{4\pi} = \frac{1}{4\pi\nu} \quad (110)$$

This establishes the connection of the noncommutative Chern-Simons action with the quantum Hall effect.

As before, gauge transformations are conjugations of  $X^i$  or  $D_i$  by arbitrary time-dependent unitary operators. In the quantum Hall fluid context they take the meaning of reshuffling the electrons. Equivalently, the  $X^i$  can be considered as coordinates of a two-dimensional fuzzy membrane,  $2\pi\theta$  playing the role of an area quantum and gauge transformations realizing area preserving diffeomorphisms. The canonical conjugate of  $X^1$  is  $P_2 = BX^2$ , and the generator of gauge transformations is

$$G = -iB[X^1, X^2] = B\theta = \frac{1}{\nu} \quad (111)$$

by virtue of (109). Since gauge transformations are interpreted as reshufflings of particles, the above has the interpretation of endowing the particles with quantum statistics of order  $1/\nu$ .

## 5.2 Quasiparticle and quasihole classical states

The classical equation (109) has a unique solution, modulo gauge (unitary) transformations, namely the unique irreducible representation of the Heisenberg algebra. Representation states can be conveniently written in a Fock basis  $|n\rangle$ ,  $n = 0, 1, \dots$ , for the ladder operators  $X^1 \pm iX^2$ ,  $|0\rangle$  representing a state of minimal spread at the origin. The classical theory has this representation as its unique state, the vacuum.

Deviations from the vacuum (109) can be achieved by introducing sources in the action [32]. A localized source at the origin has a density of the form  $\rho = \rho_0 - q\delta^2(x)$

in the continuous (commutative) case, representing a point source of particle number  $-q$ , that is, a hole of charge  $q$  for  $q > 0$ . The noncommutative analog of such a density is

$$[X^1, X^2] = i\theta(1 + q|0\rangle\langle 0|) \quad (112)$$

In the membrane picture the right-hand side of (112) corresponds to area and implies that the area quantum at the origin has been increased to  $2\pi\theta(1+q)$ , therefore piercing a hole of area  $A = 2\pi\theta q$  and creating a particle deficit  $q = \rho_0 A$ . We shall call this a quasihole state. For  $q > 0$  we find the quasihole solution of (112) as

$$X^1 + iX^2 = \sqrt{2\theta} \sum_{n=1}^{\infty} \sqrt{n+q} |n-1\rangle\langle n| \quad (113)$$

Such solutions are called noncommutative solitons [14, 51, 52].

The case of quasiparticles,  $q < 0$  is more interesting. Clearly the area quantum cannot be diminished below zero, and equations (112) and (113) cannot hold for  $-q > 1$ . The correct equation is, instead,

$$[X^1, X^2] = i\theta \left( 1 - \sum_{n=0}^{k-1} |n\rangle\langle n| - \epsilon |k\rangle\langle k| \right) \quad (114)$$

where  $k$  and  $\epsilon$  are the integer and fractional part of the quasiparticle charge  $-q$ . The solution of (114) is

$$X^1 + iX^2 = \sum_{n=0}^{k-1} z_n |n\rangle\langle n| + \sqrt{2\theta} \sum_{n=k+1}^{\infty} \sqrt{n-k-\epsilon} |n-1\rangle\langle n| \quad (115)$$

(For  $k = 0$  the first sum in (114,115) drops.) In the membrane picture,  $k$  quanta of the membrane have ‘peeled’ and occupy positions  $z_n = x_n + iy_n$  on the plane, while the rest of the membrane has a deficit of area at the origin equal to  $2\pi\theta\epsilon$ , leading to a charge surplus  $\epsilon$ . Clearly the quanta are electrons that sit on top of the continuous charge distribution. If we want all charge density to be concentrated at the origin, we must choose all  $z_n = 0$ . The above quasiparticle states for integer  $q$  are the noncommutative solitons and flux tubes that are also solutions of noncommutative gauge theory, while the quasihole states are not solutions of the noncommutative gauge theory action and have no direct analog.

Laughlin theory predicts that quasihole excitations in the quantum Hall state have their charge  $-q$  quantized in integer units of  $\nu$ ,  $q = m\nu$ , with  $m$  a positive integer. We see that the above discussion gives no hint of this quantization, while we see at least some indication of electron quantization in (114,115). Quasihole quantization will emerge in the quantum theory, as we shall see shortly, and is equivalent to a quantization condition of the noncommutative Chern-Simons term.

### 5.3 Finite number of electrons: the Chern-Simons matrix model

Describing an infinitely plane filled with electrons is not the most interesting situation. We wish to describe quantum Hall states of finite extent consisting of  $N$  electrons. Obviously the coordinates  $X^i$  of the noncommutative fluid description would have to be represented by finite  $N \times N$  matrices. The action (108), however, and the equation (109) to which it leads, are inconsistent for finite matrices, and a modified action

must be written which still captures the physical features of the quantum Hall system. Such an action exists, and leads to a matrix model truncation of the noncommutative Chern-Simons action involving a ‘boundary field’ [40]. It is

$$S = \int dt \frac{B}{2} \text{Tr} \left\{ \epsilon_{ij} (\dot{X}^i + i[A_0, X^i]) X^j + 2\theta A_0 - \omega (X^i)^2 \right\} + \Psi^\dagger (i\dot{\Psi} - A_0 \Psi) \quad (116)$$

It has the same form as the planar CS action, but with two extra terms. The first involves  $\Psi$ , a complex  $N$ -vector that transforms in the fundamental of the gauge group  $U(N)$ :

$$X^i \rightarrow U X^i U^{-1}, \quad \Psi \rightarrow U \Psi \quad (117)$$

Its action is a covariant kinetic term similar to a complex scalar fermion. We shall, however, quantize it as a boson; this is perfectly consistent, since there is no spatial kinetic term that would lead to a negative Dirac sea and the usual inconsistencies of first-order bosonic actions.

The term proportional to  $\omega$  (not to be confused with  $\theta^{-1}$ ) serves as a spatial regulator: since we will be describing a finite number of electrons, there is nothing to keep them localized anywhere in the plane. We added a confining harmonic potential which serves as a ‘box’ to keep the particles near the origin.

We can again impose the  $A_0$  equation of motion as a Gauss constraint and then put  $A_0 = 0$ . In our case it reads

$$G \equiv -iB[X^1, X^2] + \Psi\Psi^\dagger - B\theta = 0 \quad (118)$$

Taking the trace of the above equation gives

$$\Psi^\dagger \Psi = NB\theta \quad (119)$$

The equation of motion for  $\Psi$  in the  $A_0 = 0$  gauge is  $\dot{\Psi} = 0$ . So we can take it to be

$$\Psi = \sqrt{NB} |v\rangle \quad (120)$$

where  $|v\rangle$  is a constant vector of unit length. The traceless part of (118) reads

$$[X^1, X^2] = i\theta(1 - N|v\rangle\langle v|) \quad (121)$$

This is similar to (109) for the infinite plane case, with an extra projection operator. Using the residual gauge freedom under time-independent unitary transformations, we can rotate  $|v\rangle$  to the form  $|v\rangle = (0, \dots, 0, 1)$ . The above commutator then takes the form  $i\theta \text{diag}(1, \dots, 1, 1 - N)$  which is the ‘minimal’ deformation of the planar result (109) that has a vanishing trace.

In the membrane picture,  $\Psi$  is like a boundary term. Its role is to absorb the ‘anomaly’ of the commutator  $[X^1, X^2]$ , much like the case of a boundary field theory required to absorb the anomaly of a bulk (commutative) Chern-Simons field theory.

The equations of motion for  $X^i$  read

$$\dot{X}^i + \omega \epsilon_{ij} X^j = 0 \quad (122)$$

This is just a matrix harmonic oscillator. It is solved by

$$X^1 + iX^2 = e^{i\omega t} A \quad (123)$$

where  $A$  is any  $N \times N$  matrix satisfying the constraint

$$[A, A^\dagger] = 2\theta(1 - N|v\rangle\langle v|) \quad (124)$$

The classical states of this theory are given by the set of matrices  $A = X^1 + iX^2$  satisfying (124) or (121). We can easily find them by choosing a basis in which one of the  $X$ s is diagonal, say,  $X^1$ . Then the commutator  $[X^1, X^2]$  is purely off-diagonal and the components of the vector  $|v\rangle$  must satisfy  $|v_n|^2 = 1/N$ . We can use the residual  $U(1)^N$  gauge freedom to choose the phases of  $v_n$  so that  $v_n = 1/\sqrt{N}$ . So we get

$$(X^1)_{mn} = x_n \delta_{mn}, \quad (X^2)_{mn} = y_n \delta_{mn} + \frac{i\theta}{x_m - x_n} (1 - \delta_{mn}) \quad (125)$$

The solution is parametrized by the  $N$  eigenvalues of  $X^1$ ,  $x_n$ , and the  $N$  diagonal elements of  $X^2$ ,  $y_n$ .

#### 5.4 Quantum Hall ‘droplet’ vacuum

Not all solutions found above correspond to quantum Hall fluids. In fact, choosing all  $x_n$  and  $y_n$  much bigger than  $\sqrt{\theta}$  and not too close to each other, both  $X^1$  and  $X^2$  become almost diagonal; they represent  $N$  electrons scattered in positions  $(x_n, y_n)$  on the plane and performing rotational motion around the origin with angular velocity  $\omega$ . This is the familiar motion of charged particles in a magnetic field along lines of equal potential when their proper kinetic term is negligible. Quantum Hall states will form when particles coalesce near the origin, that is, for states of low energy.

To find the ground state, we must minimize the potential

$$V = \frac{B\omega}{2} \text{Tr}[(X^1)^2 + (X^2)^2] = \frac{B\omega}{2} \text{Tr}(A^\dagger A) \quad (126)$$

while imposing the constraint (121) or (124). This can be implemented with a matrix Lagrange multiplier  $\Lambda$  (essentially, solving the equations of motion including  $A_0 \equiv \Lambda$  and putting the time derivatives to zero). We obtain

$$A = [\Lambda, A], \quad \text{or} \quad X^i = i\epsilon_{ij}[\Lambda, X^j] \quad (127)$$

This is reminiscent of canonical commutation relations for a quantum harmonic oscillator, with  $\Lambda$  playing the role of the Hamiltonian. We are led to the solution

$$A = \sqrt{2\theta} \sum_{n=0}^{N-1} \sqrt{n} |n-1\rangle\langle n|, \quad \Lambda = \sum_{n=0}^{N-1} n |n\rangle\langle n|, \quad |v\rangle = |N-1\rangle \quad (128)$$

This is essentially a quantum harmonic oscillator and Hamiltonian projected to the lowest  $N$  energy eigenstates. It is easy to check that the above satisfies both (124) and (127). Its physical interpretation is clear: it represents a circular quantum Hall ‘droplet’ of radius  $\sqrt{2N\theta}$ . Indeed, the radius-squared matrix coordinate  $R^2$  is

$$R^2 = (X^1)^2 + (X^2)^2 = A^\dagger A + \frac{1}{2}[A, A^\dagger] \quad (129)$$

$$= \sum_{n=0}^{N-2} \theta(2n+1) |n\rangle\langle n| + \theta(N-1) |N-1\rangle\langle N-1| \quad (130)$$



The highest eigenvalue of  $R^2$  is  $(2N - 1)\theta$ . The particle density of this droplet is  $\rho_0 = N/(\pi R^2) \sim 1/(2\pi\theta)$  as in the infinite plane case.

The matrices  $X^i$  are known and can be explicitly diagonalized in this case. Their eigenvalues are given by the zeros of the  $N$ -th Hermite polynomial (times  $\sqrt{2\theta}$ ). In the large- $N$  limit the distribution of these zeros obeys the famous Wigner semi-circle law, with radius  $\sqrt{N}$ . Since these eigenvalues are interpreted as electron coordinates, this confirms once more the fact that the electrons are evenly distributed on a circle of radius  $\sqrt{2N\theta}$ .

### 5.5 Excited states of the model

Excitations of the classical ground state can now be considered. Any perturbation of (128) in the form of (125) is, of course, some excited state. We shall concentrate, however, on two special types of excitations.

The first is obtained by performing on  $A, A^\dagger$  all transformations generated by the infinitesimal transformation

$$A' = A + \sum_{n=0}^{N-1} \epsilon_n (A^\dagger)^n \quad (131)$$

with  $\epsilon_n$  infinitesimal complex parameters. The sum is truncated to  $N - 1$  since  $A^\dagger$  is an  $N \times N$  matrix and only its first  $N$  powers are independent. It is obvious that (124) remains invariant under the above transformation and therefore also under the finite transformations generated by repeated application of (131).

If  $A, A^\dagger$  were true oscillator operators, these would be canonical (unitary) transformations, that is, gauge transformations that would leave the physical state invariant. For the finite  $A, A^\dagger$  in (128), however, these are *not* unitary transformations and generate a new state. To understand what is that new state, examine what happens to the ‘border’ of the circular quantum Hall droplet under this transformation. This is defined by  $A^\dagger A \sim 2N\theta$  (for large  $N$ ). To find the new boundary parametrize  $A \sim \sqrt{2N\theta}e^{i\phi}$ , with  $\phi$  the polar angle on the plane and calculate  $(A^\dagger A)'$ . The new boundary in polar coordinates is

$$R'(\phi) = \sqrt{2N\theta} + \sum_{n=-N}^N c_n e^{in\phi} \quad (132)$$

where the coefficients  $c_n$  are

$$c_n = c_{-n}^* = \frac{R^n}{2} \epsilon_{n-1} \quad (n > 0), \quad c_0 = 0 \quad (133)$$

This is an arbitrary area-preserving deformation of the boundary of the droplet, truncated to the lowest  $N$  Fourier modes. The above states are, therefore, arbitrary area-preserving boundary excitations of the droplet [53, 54, 55], appropriately truncated to reflect the finite noncommutative nature of the system (the fact that there are only  $N$  electrons).

Note that on the plane there is an infinity of area-preserving diffeomorphisms that produce a specific deformation of a given curve. From the droplet point of view, however, these are all gauge equivalent since they deform the outside of the droplet (which is empty) or the inside of it (which is full and thus invariant). The finite theory that we examine has actually broken this infinite gauge freedom, since most of these

canonical transformations of  $a, a^\dagger$  do not preserve the Gauss constraint (124) when applied on  $A, A^\dagger$ . The transformations (131) pick a representative in this class which respects the constraint.

The second class of excitations are the analogs of quasihole and quasiparticle states. States with a quasihole of charge  $-q$  at the origin can be written quite explicitly in the form

$$A = \sqrt{2\theta} \left( \sqrt{q} |N-1\rangle \langle 0| + \sum_{n=1}^{N-1} \sqrt{n+q} |n-1\rangle \langle n| \right), \quad q > 0 \quad (134)$$

It can be verified that the eigenvalues of  $A^\dagger A$  are

$$(A^\dagger A)_n = 2\theta(n+q), \quad n = 0, 1, \dots, N-1 \quad (135)$$

so it represents a circular droplet with a circular hole of area  $2\pi\theta q$  at the origin, that is, with a charge deficit  $q$ . The droplet radius has appropriately swelled, since the total number of particles is always  $N$ .

Note that (134) stills respects the Gauss constraint (124) (with  $|v\rangle = |N-1\rangle$ ) *without* the explicit introduction of any source. So, unlike the infinite plane case, this model contains states representing quasiholes without the need to introduce external sources. What happens is that the hole and the boundary of the droplet together cancel the anomaly of the commutator, the outer boundary part absorbing an amount  $N+q$  and the inner (hole) boundary producing an amount  $q$ . This possibility did not exist in the infinite plane, where the boundary at infinity was invisible, and an explicit source was needed to nucleate the hole.

Quasiparticle states are a different matter. In fact, there are *no* quasiparticle states with the extra particle number localized anywhere within the droplet. Such states do not belong to the  $\nu = 1/B\theta$  Laughlin state. There are quasiparticle states with an *integer* particle number  $-q = m$ , and the extra  $m$  electrons occupying positions *outside* the droplet. The explicit form of these states is not so easy to write. At any rate, it is interesting that the matrix model ‘sees’ the quantization of the particle number and the inaccessibility of the interior of the quantum Hall state in a natural way.

Having said all that, we are now making the point that *all types of states defined above are the same*. Quasihole and quasiparticle states are nonperturbative boundary excitations of the droplet, while perturbative boundary excitations can be viewed as marginal particle states.

To clarify this point, note that the transformation (131) or (132) defining infinitesimal boundary excitations has  $2N$  real parameters. The general state of the system, as presented in (125) also depends on  $2N$  parameters (the  $x_n$  and  $y_n$ ). The configuration space is connected, so all states can be reached continuously from the ground state. Therefore, all states can be generated by exponentiating (131). This is again a feature of the finite- $N$  model: there is no sharp distinction between ‘perturbative’ (boundary) and ‘soliton’ (quasiparticle) states, each being a particular limit of the other.

## 5.6 Equivalence to the Calogero model

The model examined above should feel very familiar to Calogero model aficionados. Indeed, it is equivalent to the harmonic rational Calogero model [41, 42, 43], whose

connection to fractional statistics [44, 45] and anyons [46]-[47] has been established in different contexts. This is an integrable system of  $N$  nonrelativistic particles on the line interacting with mutual inverse-square potential and an external harmonic potential, with Hamiltonian

$$H = \sum_{n=1}^N \left( \frac{\omega}{2B} p_n^2 + \frac{B\omega}{2} x_n^2 \right) + \sum_{n \neq m} \frac{\nu^{-2}}{(x_n - x_m)^2} \quad (136)$$

In terms of the parameters of the model, the mass of the particles is  $B/\omega$  and the coupling constant of the two-body inverse-square potential is  $\nu^{-2}$ . We refer the reader to [48, 49, 50] for details on the Calogero model and its connection with the matrix model. Here we simply state the relevant results and give their connection to quantum Hall quantities.

The positions of the Calogero particles  $x_n$  are the eigenvalues of  $X^1$ , while the momenta  $p_n$  are the diagonal elements of  $X^2$ , specifically  $p_n = By_n$ . The motion of the  $x_n$  generated by the Hamiltonian (136) is compatible with the evolution of the eigenvalues of  $X^1$  as it evolves in time according to (123). So the Calogero model gives a one-dimensional perspective of the quantum Hall state by monitoring some effective electron coordinates along  $X^1$  (the eigenvalues of  $X^1$ ).

The Hamiltonian of the Calogero model (136) is equal to the potential  $V = \frac{1}{2}B\omega\text{Tr}(X^i)^2$  of the matrix model. Therefore, energy states map between the two models. The ground state is obtained by putting the particles at their static equilibrium positions. Because of their repulsion, they will form a lattice of points lying at the roots of the  $N$ -th Hermite polynomial and reproducing the semi-cycle Wigner distribution mentioned before.

Boundary excitations of the quantum Hall droplet correspond to small vibrations around the equilibrium position, that is, sound waves on the lattice. Quasiholes are large-amplitude (nonlinear) oscillations of the particles at a localized region of the lattice. For a quasihole of charge  $q$  at the center, on the average  $q$  particles near  $x = 0$  participate in the oscillation.

Finally, quasiparticles are excitations where one of the particles is isolated outside the ground state distribution (a ‘soliton’) [56]. As it moves, it ‘hits’ the distribution on one side and causes a solitary wave of net charge 1 to propagate through the distribution. As the wave reaches the other end of the distribution another particle emerges and gets emitted there, continuing its motion outside the distribution. So a quasiparticle is more or less identified with a Calogero particle, although its role, at different times, is assumed by different Calogero particles, or even by soliton waves within the ground state distribution.

Overall, we have a ‘holographic’ description of the two-dimensional quantum Hall states in terms of the one-dimensional Calogero particle picture. Properties of the system can be translated back-and-forth between the two descriptions. Further connections at the quantum level will be described in subsequent sections.

## 6 The quantum matrix Chern-Simons model

The properties of the model analyzed in the previous section are classical. The ‘states’ and ‘oscillators’ that we encountered were due to the noncommutative nature of the coordinates and were referring to the classical matrix model.

The full physical content of the model, and its complete equivalence to quantum Hall (Laughlin) states, is revealed only upon quantization. In fact, some of the most

interesting features of the states, such as filling fraction and quasihole charge quantization, manifest only in the quantum domain. This will be the subject of the present section.

### 6.1 Quantization of the filling fraction

We now come to the question of the quantization of the above matrix model. This is an important issue, since some of the relevant features of the quantum Hall state will only emerge at the quantum level. The quantization has been treated in [49]. We shall repeat here the basic arguments establishing their relevance to the quantum Hall system.

We shall use double brackets for quantum commutators and double kets for quantum states, to distinguish them from matrix commutators and  $N$ -vectors.

Quantum mechanically the matrix elements of  $X^i$  become operators. Since the lagrangian is first-order in time derivatives,  $X_{mn}^1$  and  $X_{kl}^2$  are canonically conjugate:

$$[[X_{mn}^1, X_{kl}^2]] = \frac{i}{B} \delta_{ml} \delta_{kn} \quad (137)$$

or, in terms of  $A = X^1 + iX^2$

$$[[A_{mn}, A_{kl}^\dagger]] = \frac{1}{B} \delta_{mk} \delta_{nl} \quad (138)$$

The Hamiltonian, ordered as  $\frac{1}{2}B\omega\text{Tr}A^\dagger A$ , is

$$H = \sum_{mn} \frac{1}{2} B\omega A_{mn}^\dagger A_{mn} \quad (139)$$

This is just  $N^2$  harmonic oscillators. Further, the components of the vector  $\Psi_n$  correspond to  $N$  harmonic oscillators. Quantized as bosons, their canonical commutator is

$$[[\Psi_m, \Psi_n^\dagger]] = \delta_{mn} \quad (140)$$

So the system is a priori just  $N(N+1)$  uncoupled oscillators. What couples the oscillators and reduces the system to effectively  $2N$  phase space variables (the planar coordinates of the electrons) is the Gauss law constraint (118). In writing it, we in principle encounter operator ordering ambiguities. These are easily fixed, however, by noting that the operator  $G$  is the quantum generator of unitary rotations of both  $X^i$  and  $\Psi$ . Therefore, it must satisfy the commutation relations of the  $U(N)$  algebra. The  $X$ -part is an orbital realization of  $SU(N)$  on the manifold of  $N \times N$  hermitian matrices. Specifically, expand  $X^{1,2}$  and  $A, A^\dagger$  in the complete basis of matrices  $\{1, T^a\}$  where  $T^a$  are the  $N^2 - 1$  normalized fundamental  $SU(N)$  generators:

$$X^1 = x_0 + \sum_{a=1}^{N^2-1} x_a T^a, \quad \sqrt{B}A = a_o + \sum_{a=1}^{N^2-1} a_a T^a \quad (141)$$

$x_a, a_a$  are scalar operators. Then, by (137,138) the corresponding components of  $BX^2$  are the conjugate operators  $-i\partial/\partial x_a$ , while  $a_a, a_a^\dagger$  are harmonic oscillator operators. We can write the components of the matrix commutator  $G_X = -iB[X^1, X^2]$  in  $G$  in

the following ordering

$$G_X^a = -if^{abc}x_b \frac{\partial}{\partial x_a} \quad (142)$$

$$= -i(A_{mk}^\dagger A_{nk} - A_{nk}^\dagger A_{mk}) \quad (143)$$

$$= -ia_b^\dagger f^{abc} a_c \quad (144)$$

where  $f^{abc}$  are the structure constants of  $SU(N)$ . Similarly, expressing  $G_\Psi = \Psi\Psi^\dagger$  in the  $SU(N)$  basis of matrices, we write its components in the ordering

$$G_\Psi^a = \Psi_m^\dagger T_{mn}^a \Psi_n \quad (145)$$

The operators above, with the specific normal ordering, indeed satisfy the  $SU(N)$  algebra. The expression of  $G_X^a$  in terms of  $x_a$  is like an angular momentum. The expression of  $G_\Psi^a$  in terms of the oscillators  $\Psi_i$  and of  $G_X^a$  in terms of the oscillators  $a_a$  is the well-known Jordan-Wigner realization of the  $SU(N)$  algebra in the Fock space of bosonic oscillators. Specifically, let  $R_{\alpha\beta}^a$  be the matrix elements of the generators of  $SU(N)$  in any representation of dimension  $d_R$ , and  $a_\alpha, a_\alpha^\dagger$  a set of  $d_R$  mutually commuting oscillators. Then the operators

$$G^a = a_\alpha^\dagger R_{\alpha\beta}^a a_\beta \quad (146)$$

satisfy the  $SU(N)$  algebra. The Fock space of the oscillators contains all the symmetric tensor products of  $R$ -representations of  $SU(N)$ ; the total number operator of the oscillators identifies the number of  $R$  components in the specific symmetric product. The expressions for  $G_\Psi^a$  and  $G_X^a$  are specific cases of the above construction for  $R^a$  the fundamental ( $T^a$ ) or the adjoint ( $-if^a$ ) representation respectively.

So, the traceless part of the Gauss law (118) becomes

$$(G_X^a + G_\Psi^a)|phys\rangle = 0 \quad (147)$$

where  $|phys\rangle$  denotes the physical quantum states of the model. The trace part, on the other hand, expresses the fact that the total  $U(1)$  charge of the model must vanish. It reads

$$(\Psi_n^\dagger \Psi_n - NB\theta)|phys\rangle = 0 \quad (148)$$

We are now set to derive the first nontrivial quantum mechanical implication: the inverse-filling fraction is quantized to integer values. To see this, first notice that the first term in (148) is nothing but the total number operator for the oscillators  $\Psi_n$  and is obviously an integer. So we immediately conclude that  $NB\theta$  must be quantized to an integer.

However, this is not the whole story. Let us look again at the  $SU(N)$  Gauss law (147). It tells us that physical states must be in a singlet representation of  $G^a$ . The orbital part  $G_X^a$ , however, realizes only representations arising out of products of the adjoint, and therefore it contains only irreps whose total number of boxes in their Young tableau is an integer multiple of  $N$ . Alternatively, the  $U(1)$  and  $Z_N$  part of  $U$  is invisible in the transformation  $X^i \rightarrow UX^iU^{-1}$  and thus the  $Z_N$  charge of the operator realizing this transformation on states must vanish. (For instance, for  $N = 2$ ,  $G^a$  is the usual orbital angular momentum in 3 dimensions which cannot be half-integer.)

Since physical states are invariant under the sum of  $G_X$  and  $G_\Psi$ , the representations of  $G_\Psi$  and  $G_X$  must be conjugate to each other so that their product contain

the singlet. Therefore, the irreps of  $G_\Psi$  must also have a number of boxes which is a multiple of  $N$ . The oscillator realization (148) contains all the symmetric irreps of  $SU(N)$ , whose Young tableau consists of a single row. The number of boxes equals the total number operator of the oscillators  $\Psi_n^\dagger \Psi_n$ . So we conclude that  $NB\theta$  must be an integer multiple of  $N$  [49], that is,

$$B\theta = \frac{1}{\nu} = k, \quad k = \text{integer} \quad (149)$$

The above effect has a purely group theoretic origin. The same effect, however, can be recovered using topological considerations, by demanding invariance of the quantum action  $\exp(iS)$  under gauge  $U(N)$  transformations with a nontrivial winding in the temporal direction [49]. This is clearly the finite- $N$  counterpart of the level quantization for the noncommutative Chern-Simons term as exposed in a previous section, namely  $4\pi\lambda = \text{integer}$ . By (110) this is equivalent to (149).

By reducing the model to the dynamics of the eigenvalues of  $X^1$  we recover a quantum Calogero model with Hamiltonian

$$H = \sum_{n=1}^N \left( \frac{\omega}{2B} p_n^2 + \frac{B\omega}{2} x_n^2 \right) + \sum_{n \neq m} \frac{k(k+1)}{(x_n - x_m)^2} \quad (150)$$

Note the shift of the coupling constant from  $k^2$  to  $k(k+1)$  compared to the classical case. This is a quantum reordering effect which results in the shift of  $\nu^{-1}$  from  $k$  to  $k+1 \equiv n$ . The above model is, in fact, perfectly well-defined even for fractional values of  $\nu^{-1}$ , while the matrix model that generated it requires quantization. This is due to the fact that, by embedding the particle system in the matrix model, we have augmented its particle permutation symmetry  $S_N$  to general  $U(N)$  transformations; while the smaller symmetry  $S_N$  is always well-defined, the larger  $U(N)$  symmetry becomes anomalous unless  $\nu^{-1}$  is quantized.

## 6.2 Quantum states

We can now examine the quantum excitations of this theory. The quantum states of the model are simply states in the Fock space of a collection of oscillators. The total energy is the energy carried by the  $N^2$  oscillators  $A_{mn}$  or  $a_a$ . We must also impose the constraint (147) and (148) on the Fock states. Overall, this becomes a combinatorics group theory problem which is in principle doable, although quite tedious.

Fortunately, we do not need to go through it here. The quantization of this model is known and achieves its most intuitive description in terms of the states of the corresponding Calogero model. We explain how.

Let us work in the  $X^1$  representation,  $X^2$  being its canonical momentum. Writing  $X^1 = U\Lambda_1 U^{-1}$  with  $\Lambda = \text{diag}\{x_i\}$  being its eigenvalues, we can view the state of the system as a wavefunction of  $U$  and  $x_n$ . The gauge generator  $G_X^a$  appearing in the Gauss law (147) is actually the conjugate momentum to the variables  $U$ . Due to the Gauss law, the angular degrees of freedom  $U$  are constrained to be in a specific angular momentum state, determined by the representation of  $SU(N)$  carried by the  $\Psi_n$ . From the discussion of the previous section, we understand that this is the completely symmetric representation with  $nN = N/\nu$  boxes in the Young tableau. So the dynamics of  $U$  are completely fixed, and it suffices to consider the states of the eigenvalues. These are described by the states of the quantum Calogero model.

The Hamiltonian of the Calogero model corresponds to the matrix potential  $V = \frac{1}{2}B\omega\text{Tr}(X^i)^2$ , which contains all the relevant information for the system.

Calogero energy eigenstates are expressed in terms of  $N$  positive, integer ‘quasi-occupation numbers’  $n_j$  (quasiumbers, for short), with the property

$$n_j - n_{j-1} \geq n = \frac{1}{\nu}, \quad j = 1, \dots, N \quad (151)$$

In terms of the  $n_j$  the spectrum becomes identical to the spectrum of  $N$  independent harmonic oscillators

$$E = \sum_{j=1}^N E_j = \sum_{j=1}^N \omega \left( n_j + \frac{1}{2} \right) \quad (152)$$

The constraint (151) means that the  $n_j$  cannot be packed closer than  $n = \nu^{-1}$ , so they have a ‘statistical repulsion’ of order  $n$ . For filling fraction  $\nu = 1$  these are ordinary fermions, while for  $\nu^{-1} = n > 1$  they behave as particles with an enhanced exclusion principle.

The scattering phase shift between Calogero particles is  $\exp(i\pi/\nu)$ . So, in terms of the phase that their wavefunction picks upon exchanging them, they look like fermions for odd  $n$  and bosons for even  $n$  [44]. Since the underlying particles (electrons) must be fermions, we should pick  $n$  odd.

The energy ‘eigenvalues’  $E_j$  are the quantum analogs of the eigenvalues of the matrix  $\frac{1}{2}B\omega(X^i)^2$ . The radial positions  $R_j$  are determined by

$$\frac{1}{2}B\omega R_j^2 = E_j \quad \rightarrow \quad R_j^2 = \frac{2n_j + 1}{B} \quad (153)$$

So the quasiumbers  $2n_j + 1$  determine the radial positions of electrons. The ground state values are the smallest non-negative integers satisfying (151)

$$n_{j,gs} = n(j-1), \quad j = 1, \dots, N \quad (154)$$

They form a ‘Fermi sea’ but with a density of states dilated by a factor  $\nu$  compared to standard fermions. This state reproduces the circular quantum Hall droplet. Its radius maps to the Fermi level,  $R \sim \sqrt{(2n_{N,gs} + 1)/B} \sim \sqrt{2N\theta}$ .

Quasiparticle and quasihole states are identified in a way analogous to particles and holes of a Fermi sea. A quasiparticle state is obtained by peeling a ‘particle’ from the surface of the sea (quasiumber  $n_{N,gs}$ ) and putting it to a higher value  $n'_N > n(N-1)$ . This corresponds to an electron in a rotationally invariant state at radial position  $R' \sim \sqrt{2(n'_N + 1)/B}$ . Successive particles can be excited this way. The particle number is obviously quantized to an integer (the number of excited quasiumbers) and we can only place them outside the quantum Hall droplet.

Quasiholes are somewhat subtler: they correspond to the minimal excitations of the ground state *inside* the quantum Hall droplet. This can be achieved by leaving all quasiumber  $n_j$  for  $j \leq k$  unchanged, and increasing all  $n_j$ ,  $j > k$  by one

$$n_j = n(j-1) \quad j \leq k \quad (155)$$

$$= n(j-1) + 1 \quad k < j \leq N \quad (156)$$

This increases the gap between  $n_k$  and  $n_{k+1}$  to  $n+1$  and creates a minimal ‘hole.’

This hole has a particle number  $-q = -1/n = -\nu$ . To see it, consider removing a particle altogether from quasiumber  $n_k$ . This would create a gap of  $2n$  between  $n_{k-1}$

and  $n_{k+1}$ . The extra gap  $n$  can be considered as arising out of the formation of  $n$  holes (increasing  $n_j$  for  $j \geq k$   $n$  times). Thus the absence of a particle corresponds to  $n$  holes. We therefore obtain the important result that the quasihole charge is naturally quantized to units of

$$q_h = \nu = \frac{1}{n} \quad (157)$$

in accordance with Laughlin theory.

We conclude by stressing once more that there is no fundamental distinction between particles and holes for finite  $N$ . A particle can be considered as a nonperturbative excitation of many holes near the Fermi level, while a hole can be viewed as a coherent state of many particles of minimal excitation.

### 6.3 Final remarks on the matrix model

The quantization of the inverse filling fraction and, importantly, the quasihole charge quantization emerged as quantum mechanical consequences of this model. The quantizations of the two parameters had a rather different origin. We can summarize here the basic meaning of each:

Quantization of the inverse filling fraction is basically angular momentum quantization. The matrix commutator of  $[X_1, X_2]$  is an orbital angular momentum in the compact space of the angular parameters of the matrices, and it must be quantized. Alternatively (and equivalently), it can be understood as a topological quantization condition due to a global gauge anomaly of the model.

Quantization of the quasihole charge, on the other hand, is nothing but harmonic oscillator quantization. Quasiholes are simply individual quanta of the oscillators  $A_{mn}$ . The square of the radial coordinate  $R^2 = (X^1)^2 + (X^2)^2$  is basically a harmonic oscillator.  $\sqrt{B}X_1$  and  $\sqrt{B}X_2$  are canonically conjugate, so the quanta of  $R^2$  are  $2/B$ . Each quantum increases  $R^2$  by  $2/B$  and so it increases the area by  $2\pi/B$ . This creates a charge deficit  $q$  equal to the area times the ground state density  $q = (2\pi/B) \cdot (1/2\pi\theta) = 1/\theta B = \nu$ . So the fundamental quasihole charge is  $\nu$ .

An important effect, which can be both interesting and frustrating, is the quantum shift in the effective value of the inverse filling fraction from  $k$  to  $n = k + 1$ . This is the root of the famous fermionization of the eigenvalues of the matrix model in the singlet sector ( $k = 0$ ). Its presence complicates some efforts to reproduce layered quantum Hall states, as it frustrates the obvious charge density counting.

There are obviously many questions that we left untouched, some of them already addressed and some still open [57]-[64]. Their list includes the description of Hall states with spin, the treatment of cylindrical, spherical or toroidal space topologies, the description of states with nontrivial filling fraction, the exact mapping between quantities of physical interest in the two descriptions, the inclusion of electron interactions etc. The interested reader is directed to the numerous papers in the literature dealing with these issues. In the concluding section we prefer to present an alternative of the noncommutative fluid description of quantum many-body states.

## 7 The noncommutative Euler picture and Bosonization

In the previous sections we reviewed the noncommutative picture of the Lagrange formulation of fluids and its use in the quantum Hall effect. The Euler formulation, on the other hand, was peculiar in that it afforded a fully commutative description was possible, leading to the Seiberg-Witten map.



This, however, is not the only possibility. Indeed, we saw that there were *two* potential descriptions for the density of the fluid, one inherently commutative (94) and one inherently noncommutative (91). Although the commutative one was adopted, one could just as well work with the noncommutative one, expecting to recover the standard Euler description only at the commutative limit. As it turns out, this is a very natural description of fluids consisting of fermions. Since the noncommutative density is an inherently bosonic field, it affords a description of fermionic systems in terms of bosonic field variables, naturally leading to bosonization.

### 7.1 Density description of fermionic many-body systems

The starting point will be a system of  $N$  non-interacting fermions in  $D = 1$  spatial dimensions. The restriction of the dimensionality of space at this point is completely unnecessary and inconsequential, and is imposed only for conceptual and notational simplification and easier comparison with previous sections. In fact, much of the formalism will not even make specific reference to the dimensionality of space.

We shall choose our fermions to be noninteracting and carrying no internal degrees of freedom such as spin, color etc. (there is no conflict with the spin-statistics theorem in this first-quantized, many-body description). Again, this is solely for convenience and to allow us to focus on the main conceptual issue of their fluid description rather than other dynamical questions. The only remaining physical quantity is the single-particle Hamiltonian defining their dynamics, denoted  $H_{\text{sp}}(x, p)$ . Here  $x, p$  are single-particle coordinate and momentum operators, together forming a ‘noncommutative plane’, with the role of  $\theta$  played by  $\hbar$  itself:

$$[x, p]_{\text{sp}} = i\hbar \quad (158)$$

The subscript sp will be appended to single-particle operators or relations (except  $x$  and  $p$ ) to distinguish them from upcoming field theory quantities.

Single-particle states are elements of the irreducible representation of the above Heisenberg commutator. A basis would be the eigenstates  $|n\rangle$  of  $H_{\text{sp}}$  corresponding to eigenvalues  $E_n$  (assumed nondegenerate for simplicity). The states of the  $N$ -body system, on the other hand, are fully antisymmetrized elements of the  $N$ -body Hilbert space consisting of  $N$  copies of the above space. They can be expressed in a Fock description in terms of the occupation number basis  $N_n = 0, 1$  for each single particle level. The ground state, in particular, is the state  $|1, \dots, 1, 0, \dots\rangle$  with the  $N$  lowest levels occupied by fermions.

An alternative description, however, working with a *single* copy of the above space is possible, in terms of a single-particle density-like function [65, 66]. Specifically, define the (hermitian) single-particle operator  $\rho$  whose eigenvalues correspond to the the occupation numbers  $N_i = 0, 1$  of a set of  $N$  specific single-particle states:

$$\rho = \sum_{n=1}^N |\psi_n\rangle\langle\psi_n| \quad (159)$$

Clearly  $\rho$  is a good description of the  $N$ -body fermion system whenever the fermions occupy  $N$  single-particle states. The ground state  $\rho_0$ , in particular, is such a state and would correspond to

$$\rho_0 = \sum_{n=1}^N |n\rangle\langle n| \quad (160)$$

Due to the Schrödinger evolution of the single-particle states  $|n\rangle$ , the operator  $\rho$  satisfies the evolution equation

$$i\hbar\dot{\rho} = [H_{\text{sp}}, \rho]_{\text{sp}} \quad (161)$$

This description has several drawbacks. It is obviously limited from the fact that it can describe only ‘factorizable’ states, that is, basis states in some appropriate Fock space, but not their linear combinations (‘entangled’ states). This is serious, as it violates the quantum mechanical superposition principle, and makes it clear that this cannot be a full quantum description of the system. Further, the operator  $\rho$  must be a projection operator with exactly  $N$  eigenvalues equal to one and the rest of them vanishing, which can be written as the algebraic constraint

$$\rho^2 = \rho, \quad \text{Tr}\rho = N \quad (162)$$

So  $\rho$  is similar to the density matrix, except for its trace.

In spite of the above, we shall see that this is a valid starting point for a full description of the many-body quantum system in a second-quantized picture. To give  $\rho$  proper dynamics, we must write an action that leads to the above equations (evolution plus constraints) in a canonical setting. The simplest way to achieve this is by ‘solving’ the constraint in terms of a unitary field  $U$  as:

$$\rho = U^{-1}\rho_0U \quad (163)$$

with  $\rho_0$  the ground state. Any  $\rho$  can be expressed as above,  $U$  being the unitary operator mapping the first  $N$  energy eigenstates to the actual single-particle states entering the definition of  $\rho$ . An appropriate action for  $U$  is

$$S = \int dt \text{Tr} \left( i\hbar\rho_0\dot{U}U^{-1} - U^{-1}\rho_0UH_{\text{sp}} \right) \quad (164)$$

It is easy to check that it leads to (161) for (163). Note that the first term in the action is a first-order kinetic term, defining a canonical one-form. The matrix elements of  $U$ , therefore, encode both coordinates and momenta and constitute the full phase space variable of the system. The Poisson brackets of  $U$  and, consequently,  $\rho$  can be derived by inverting the above canonical one-form. The result is that the matrix elements  $\rho_{mn}$  of  $\rho$  have Poisson brackets

$$\{\rho_{m_1n_1}, \rho_{m_2n_2}\} = \frac{1}{i\hbar} (\rho_{m_1n_2}\delta_{m_2n_1} - \rho_{m_2n_1}\delta_{m_1n_2}) \quad (165)$$

The second term in the action is the Hamiltonian  $H = \text{Tr}(\rho H_{\text{sp}})$  and represents the sum of the energy expectation values of the  $N$  fermions.

## 7.2 The correspondence to a noncommutative fluid

It should be clear the the above description essentially defines a noncommutative fluid. Indeed, the operators  $U$  and  $\rho$  act on the Heisenberg Hilbert space and can be expressed in terms of the fundamental operators  $x, p$ . As such, they are noncommutative fields. The constraint for  $\rho$  is the noncommutative version of the relation  $f^2 = f$  defining the characteristic function of a domain. We can, therefore, visualize  $\rho$  as a ‘droplet’ of a noncommutative fluid that fills a ‘domain’ of the noncommutative plain with a droplet ‘hight’ equal to 1. The actual density of the fluid is fixed by the

integration formula on the noncommutative plane, assigning an area of  $2\pi\hbar$  to each state on the Hilbert space. So the value of the density inside the droplet becomes  $1/2\pi\hbar$ .

A similar picture is obtained by considering the classical ‘symbol’ of the above operator, using the Weyl-ordering mapping. The corresponding commutative function represents a droplet with a fuzzy boundary (the field drops smoothly from 1 to 0, and can even become negative at some points), but the bulk of the droplet and its exterior are at constant density (0 or 1).

As one should expect, this is the value of the density of states on phase space according to the semiclassical quantization condition assigning one quantum state per phase space area  $h = 2\pi\hbar$ . The above description is the quantum, fuzzy, non-commutative analog of the classical phase space density. According to the Liouville theorem, a collection of particles with some density on the phase space evolves in an area-preserving way, so a droplet of constant density evolves into a droplet of different shape but the same constant density [67].

The ground state  $\rho_0$  corresponds to a droplet filling a ‘lake’ in phase space in which the classical value of the single particle energy satisfies

$$H_{\text{sp}}(x, p) \leq E_F \quad (166)$$

This ensures the minimal energy for the full state. The boundary of the droplet is at the line defined by the points  $H_{\text{sp}} = E_F$ , the highest energy of any single particle. This is the Fermi energy.

The unitary transformation  $U$  maps to a ‘star-unitary’ commutative function satisfying  $U * U^* = 1$ . One could think that in the commutative (classical) limit it becomes a phase,  $U = \exp[i\phi(x, p)]$ . This, however, is not necessarily so.  $U$  enters into the definition of  $\rho$  only through the adjoint action  $\rho = U^* * \rho_0 * U$ . If  $U$  became a phase in the commutative limit, it would give  $\rho = \rho_0$  (upon mapping star products to ordinary products), creating no variation. The trick is that  $U(x, p)$  can contain terms of order  $\hbar^{-1}$ : since the star-products in the definition of  $\rho$  in terms of  $U$  reproduce  $\rho_0$  plus terms of order  $\hbar$ , the overall result will be of order to a finite function in this limit; its action on  $\rho_0$ , however, is finite and defines a canonical transformation, changing the shape of the droplet. Overall, we have a correspondence with a fuzzy, incompressible phase space fluid in the density (Euler) description.

### 7.3 Quantization and the full many-body correspondence

What makes this description viable and useful is that it reproduces the *full Hilbert space* of the  $N$  fermions upon quantization.

The easiest way to see this is to notice that the action (164) is of the Kirillov-Kostant-Souriau form for the group of unitary transformations on the Hilbert space. For concreteness, we may introduce a cutoff and truncate the Hilbert space to the  $K$  first energy levels  $K \gg N$ . Then the above becomes the KKS action for the group  $U(K)$ . Its properties and quantization are fully known, and we summarize the basic points.

Both  $\rho = u^{-1}\rho_0U$  and the action (164) are invariant under time-dependent transformations

$$U(t) \rightarrow V(t)U(t), \quad [\rho_0, V(t)] = 0 \quad (167)$$

for any unitary operator ( $K \times K$  unitary matrix) commuting with  $\rho_0$ . This means that the corresponding ‘diagonal’ degrees of freedom of  $U$  are redundant and correspond to

a gauge invariance of the description in terms of  $U$ . This introduces a Gauss law as well as a ‘global gauge anomaly’ for the action that requires a quantization condition, akin to the magnetic monopole quantization or level quantization for the Chern-Simons term. The end result is:

- The eigenvalues of the constant matrix  $\rho_0$  must be integers for a consistent quantization.

On the other hand, the classical Poisson brackets for  $\rho$  (165) become, upon quantization,

$$[[\rho_{m_1 n_1}, \rho_{m_2 n_2}]] = \rho_{m_1 n_2} \delta_{m_2 n_1} - \rho_{m_2 n_1} \delta_{m_1 n_2} \quad (168)$$

where we used, again, double brackets for quantum commutators to distinguish from matrix (single-particle) commutators. The above is nothing but the  $U(K)$  algebra in a ‘cartesian’ basis (notice how  $\hbar$  has disappeared). The quantum Hilbert space, therefore, will form representations of  $U(K)$ . The Gauss law, however, imposes constraints on what these can be. The end result is:

- The quantum states form an irreducible representation of  $U(K)$  determined by a Young tableau with the number of boxes in each row corresponding to the eigenvalues of  $\rho_0$ .

In our case, the eigenvalues are  $N$  1s and  $K - N$  0s, already properly quantized. So the Young tableau corresponds to a single column of  $N$  boxes; that is, the  $N$ -fold fully antisymmetric representation of  $U(K)$ .

This is exactly the Hilbert space of  $N$  fermions on  $K$  single-particle states! The dimensionality of this representation is

$$D = \frac{K!}{N!(K-N)!} \quad (169)$$

matching the number quantum states of  $N$  fermions in  $K$  levels. The matrix elements of the operator  $\rho_{mn}$  in the above representation can be realized in a Jordan-Wigner construction involving  $K$  fermionic oscillators  $\Psi_n$ , as

$$\rho_{mn} = \Psi_n^\dagger \Psi_m \quad (170)$$

satisfying the constraint

$$\sum_{n=1}^K \Psi_n^\dagger \Psi_n = N \quad (171)$$

This  $\Psi$  is essentially the second-quantized fermion field, the above relation being the constraint to the  $N$ -particle sector. The quantized Hamiltonian operator for  $\rho$  in this realization becomes

$$H = \text{Tr}(\rho H_{\text{sp}}) = \sum_{m,n} \Psi_m^\dagger (H_{\text{sp}})_{mn} \Psi_n \quad (172)$$

and thus also corresponds to the second-quantized many-body Hamiltonian. Overall, this becomes a *complete* description of the many-body fermion system in terms of a quantized noncommutative density field  $\rho$  or, equivalently, the unitary noncommutative field  $U$ .

#### 7.4 Bosonization and beyond...

The above also constitutes an exact bosonization of the fermion system. Indeed, the fields  $\rho$  or  $U$  are bosonic, so they afford a description of fermions without use of Grassman variables. The price to pay is the increase of dimensionality (two phase space

rather than one space dimensions) and the noncommutative nature of the classical  $\rho$ -dynamics, even before quantization.

The correspondence to traditional bosonization is achieved through the Seiberg-Witten map on the field  $U$ . We shall not enter into any detail here, but the upshot of the story is that the action (164) maps to the (commutative) action of a one-dimensional chiral boson under this map. The corresponding space derivative of the field is an abelian ‘current’ that maps to the boundary of the classical fluid droplet, which parametrizes the full shape of the fluid. Overall this recovers standard abelian bosonization results [68] in the noncommutative hydrodynamic setting.

Generalizations to particles carrying internal degrees of freedom are possible and lead to the Wess-Zumino-Witten action for nonabelian bosonization [69]. Most intriguingly, the above discussion can be exported to higher dimensions, leading to bosonization of higher-dimensional fermion systems through a noncommutative field (fluid) theory [73]. Extensions and applications of the above issues are the subject of ongoing investigation.

This was a lightning review of the more recent and current aspects of noncommutative fluids and their uses in many-body systems. There is a lot more to learn and do. If some of the readers are inspired and motivated into further study or research in this subject, then this narrative has served its purpose. We shall stop here.

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