Anyons and Lowest Landau Level Anyons

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Abstract. Intermediate statistics interpolating from Bose statistics to Fermi statistics are allowed in two dimensions. This is due to the particular topology of the two dimensional configuration space of identical particles, leading to non trivial braiding of particles around each other. One arrives at quantum many-body states with a multivalued phase factor, which encodes the anyonic nature of particle windings. Bosons and fermions appear as two limiting cases. Gauging away the phase leads to the so-called anyon model, where the charge of each particle interacts "à la Aharonov-Bohm" with the fluxes carried by the other particles. The multivaluedness of the wave function has been traded off for topological interactions between ordinary particles. An alternative Lagrangian formulation uses a topological Chern-Simons term in 2+1 dimensions. Taking into account the short distance repulsion between particles leads to an Hamiltonian well defined in perturbation theory, where all perturbative divergences have disappeared. Together with numerical and semi-classical studies, perturbation theory is a basic analytical tool at disposal to study the model, since finding the exact N-body spectrum seems out of reach (except in the 2-body case which is solvable, or for particular classes of N-body eigenstates which generalize some 2-body eigenstates). However, a simplification arises when the anyons are coupled to an external homogeneous magnetic field. In the case of a strong field, by projecting the system on its lowest Landau level (LLL, thus the LLL-anyon model), the anyon model becomes solvable, i.e. the classes of exact eigenstates alluded to above provide for a complete interpolation from the LLL-Bose spectrum to the LLL-Fermi spectrum. Being a solvable model allows for an explicit knowledge of the equation of state and of the mean occupation number in the LLL, which do indeed interpolate from the Bose to the Fermi cases. It also provides for a combinatorial interpretation of LLL-anyon braiding statistics in terms of occupation of single particle states. The LLL-anyon model might also be relevant experimentally: a gas of electrons in a strong magnetic field is known to exhibit a quantized Hall conductance, leading to the integer and fractional quantum Hall effects. Haldane/exclusion statistics, introduced to describe FQHE edge excitations, is a priori different from anyon statistics, since it is not defined by braiding considerations, but rather by counting arguments in the space of available states. However, it has been shown to lead to the same kind of thermodynamics as the LLL-anyon thermodynamics (or, in other words, the LLL-anyon model is a microscopic quantum mechanical realization of Haldane's statistics). The one dimensional Calogero model is also shown to have the same kind of thermodynamics as the LLL-anyons thermodynamics. This is not a coincidence: the LLL-anyon model and the Calogero model are intimately

related, the latter being a particular limit of the former. Finally, on the purely combinatorial side, the minimal difference partition problem -partition of integers with minimal difference constraints on their parts- can also be mapped on an abstract exclusion statistics model with a constant one-body density of states, which is neither the LLL-anyon model nor the Calogero model.

1   Introduction

Quantum statistics, which is concerned with quantum many-body wavefunctions of identical particles, has a long history going back to Bose and Fermi. The concept of statistics originates at the classical level in the Gibbs paradox, which is solved by means of the indiscernability postulate for identical particles. At the quantum level, the usual reasoning shows that only two types of statistics can exist, bosonic or fermionic. Indeed, since

- interchanging the positions of two identical particles can only amount to multiplying their 2-body wavefunction by a phase factor,

- a double exchange puts back the particles at their original position,

- and one usually insists on the univaluedness of the wavefunction,

this phase factor can be only 1 (boson) or -1 (fermion).

However, non trivial phase factor should be possible, since wavefunctions are anyway defined up to a phase. The configuration space of two, or more generally, N identical particles has to be defined cautiously [1]: denoting by C the configuration space of a single particle ($C = R^2$ for particles in the two-dimensional plane, $d = 2$), the configuration space of N particles should be of the type C^N/S_N, where $C * C * ... * C = C^N$ and S_N is the permutation group for N identical particles. Quotienting by S_N takes into account the identity of the particles which implies that one cannot distinguish between two configurations related by an operation of the permutation group. One should also subtract from C^N the diagonal of the configuration space D_N, i.e. any configurations where two or more particles coincide. The reason is, having in mind Fermi statistics, that the Pauli exclusion principle should be enforced in some way. A more precise argument is to have a valid classification of paths in the N-particle configuration space, which would be ambiguous if two or more particles coinciding at some time is allowed (since they are identical, did they cross each other, or did they scatter off each other ?). It follows that the configuration space of N identical particles should be

$$\tilde{C}_N = \frac{C^N - D_N}{S_N} \tag{1}$$

Note that on this configuration space, a fermionic wavefunction is multivalued (two values 1 and -1), so there is no reason not to allow more general multivaluedness. Here come some topological arguments, which allow to distinguish between $d = 2$ and $d > 2$, and, as we will see later, which can be related to spin considerations. In 2 dimensions, C^N is multiply connected and its topology is non trivial: it is not possible to shrink a path of a particle encircling another particle, due to the topological obstruction materialized by the latter. It follows that $\tilde{C}_N$ is multiply connected. This is not the case in dimensions higher than 2, where C^N is simply

connected, meaning that all paths made by a particle can be continuously deformed into each other, i.e. one cannot distinguish the interior from the exterior of a closed path of a particle around other particles.

These arguments imply that the equivalent classes of paths (first homotopy group) in $\tilde{C}_N$ are, when $d = 2$, in one-to-one correspondence with the elements of the braid group

$$\Pi_1(\tilde{C}_N) = B_N \qquad (2)$$

whereas, when $d > 2$, they are in one-to-one correspondence with the elements of the permutation group

$$\Pi_1(\tilde{C}_N) = S_N \qquad (3)$$

The braid group generators σ_i interchange the position of particle i with particle $i + 1$. This operation can be made in an anti-clockwise manner (σ_i) or a clockwise manner (σ_i^{-1}). Each braiding of N particles consists of a sequence of interchanges of pairs of neighboring particles via the σ_i's and the σ_i^{-1}'s, with $i = 1, 2, ..., N - 1$. The braid group relations list the equivalent braiding, i.e. braiding that can be continuously deformed one into the other without encountering a topological obstruction

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}; \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{when} \quad |i - j| > 2 \qquad (4)$$
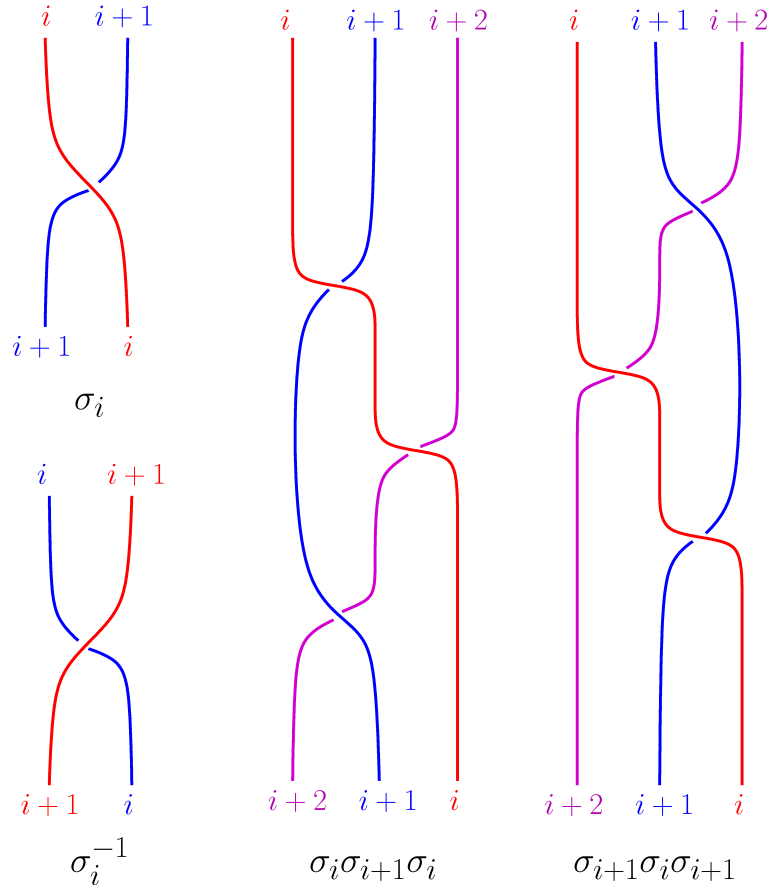

Figure 1: The braid group generators and their defining relations.

Saying that $d = 2$ is different from $d > 2$ is nothing but recognizing that $\sigma_i \neq \sigma_i^{-1}$ when $d = 2$, whereas $\sigma_i = \sigma_i^{-1}$ when $d > 2$ (σ_i can be continuously

deformed into σ_i^{-1} when particles i and $i+1$ are not stuck in a plane). It follows that when $d > 2$, the braid generators σ_i's defined by (4) with the additional constraint $\sigma_i = \sigma_i^{-1}$ are the permutation group generators.

Note also that the $d = 2$ paradigm, $\sigma_i \neq \sigma_i^{-1}$, hints at an orientation of the plane, a hallmark of the presence of some sort of magnetic field. This point will become apparent in the Aharonov-Bohm formulation of the anyon model.

The fact that C^N is multiply connected when $d = 2$ and not when $d > 2$ can also be related to the rotation group $O(d)$, and thus to some spin-statistics considerations [2]. When $d > 2$, the rotation group is doubly connected, $[\Pi_1(O(d)) = Z_2]$, its universal covering, for example when $d = 3$, is $SU(2)$, which allows for either integer or half integer angular momentum states, that is to say either single valued or double valued representations of the rotation group. On the other hand, when $d = 2$, the rotation group is abelian and infinitely connected $[\Pi_1(O(2)) = Z]$, its universal covering is the real line, that is to say arbitrary angular momenta are possible, and therefore multivalued representations. One can see here a hint about the spin-statistics connection, where statistics and spin are trivial (Bose-Fermi statistics, integer-half integer spin) when $d > 2$, and not when $d = 2$.

Let us consider the simple one-dimensional irreducible representation of the braid group, which amounts to a common phase factor $\exp(-i\pi\alpha)$ for each generator σ (and thus $\exp(i\pi\alpha)$ to σ^{-1}). It means that a non trivial phase has been associated with the winding of particle i around particle $i + 1$. Higher dimensional representations (quantum vector states) are possible -one speaks of non abelian anyons, in that case not only a non trivial phase materializes during a winding, but also the direction of the vector state in the Hilbert space is affected- but they will not be discussed here (even though they might play a role in the discussion of certain FQHE fractions [3], and, in a quite different perspective, in the definition of topologically protected fault-tolerant quantum computers [4]).

Clearly, when $d > 2$, $\sigma_i = \sigma_i^{-1}$ implies $\alpha = 0$ or $\alpha = 1$, i.e. Bose or Fermi statistics (an interchange leaves the wavefunction unchanged or affected by a minus sign).

From now on let us concentrate on $d = 2$ and denote the free many-body wavefunction of N identical particles by $\psi'(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$. Indeed, statistics should be defined for free particles with Hamiltonian

$$H'_N = \sum_{i=1}^{N} \frac{\vec{p}_i^2}{2m} \tag{5}$$

and special boundary conditions on the wavefunction, as in the Bose case (symmetric boundary condition) and the Fermi case (antisymmetric boundary condition). As already said, $\psi'(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ is affected by a phase $\exp(-i\pi\alpha)$ when particles i and $i+1$ are interchanged: one can encode this non trivial exchange property by defining

$$\psi'(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = \exp(-i\alpha \sum_{i<j} \theta_{ij}) \psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) \tag{6}$$

where $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ is a regular wavefunction, say bosonic by convention, and θ_{ij} is the angle between the vector $\vec{r}_j - \vec{r}_i \equiv \vec{r}_{ij}$ and a fixed direction in the plane. Indeed interchanging i with j amounts to $\theta_{ij} \to \theta_{ij} \pm \pi$, which altogether with the bosonic symmetry of $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$, leads to

$$\psi'(\vec{r}_1, \vec{r}_2, ...\vec{r}_j, ...\vec{r}_i, ..., \vec{r}_N) = \exp(\mp i\pi\alpha)\psi'(\vec{r}_1, \vec{r}_2, ...\vec{r}_i, ...\vec{r}_j, ..., \vec{r}_N) \tag{7}$$

By the above bosonic convention for $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$, the statistical parameter α even (odd) integer corresponds to Bose (Fermi) statistics. It is defined modulo 2, since two quanta of flux can always be gauged away by a regular gauge transformation while preserving the symmetry of the wavefunctions in the Bose or Fermi systems. Indeed, (6) can be interpreted as a gauge transformation. Let us compute the resulting Hamiltonian H_N acting on $\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$

$$H_N = \sum_{i=1}^{N} \frac{1}{2m}(\vec{p}_i - \vec{A}(\vec{r}_i))^2 \tag{8}$$

where

$$\vec{A}(\vec{r}_i) = \alpha \vec{\partial}_i (\sum_{k<l} \theta_{kl}) = \alpha \sum_{j, j \neq i} \frac{\vec{k} \wedge \vec{r}_{ij}}{r_{ij}^2} \tag{9}$$

is the statistical potential vector associated with the multivalued phase (the gauge parameter). The free multivalued wavefunction has been traded off for a regular bosonic wavefunction with topological singular magnetic interactions. The statistical potential vector (9) can be viewed as the Aharonov-Bohm (A-B) potential vector that particle i carrying a charge e would feel due to the flux tube ϕ carried by the other particles, with e and ϕ related to the statistical parameter α by $\alpha = e\phi/(2\pi) = \phi/\phi_0$ ($\phi_0 = 2\pi/e$ is the flux quantum in units $\hbar = 1$). The resulting composite charge-flux picture is known under the name of anyon model [5] since it describes particles with "any" (any-on) statistics.
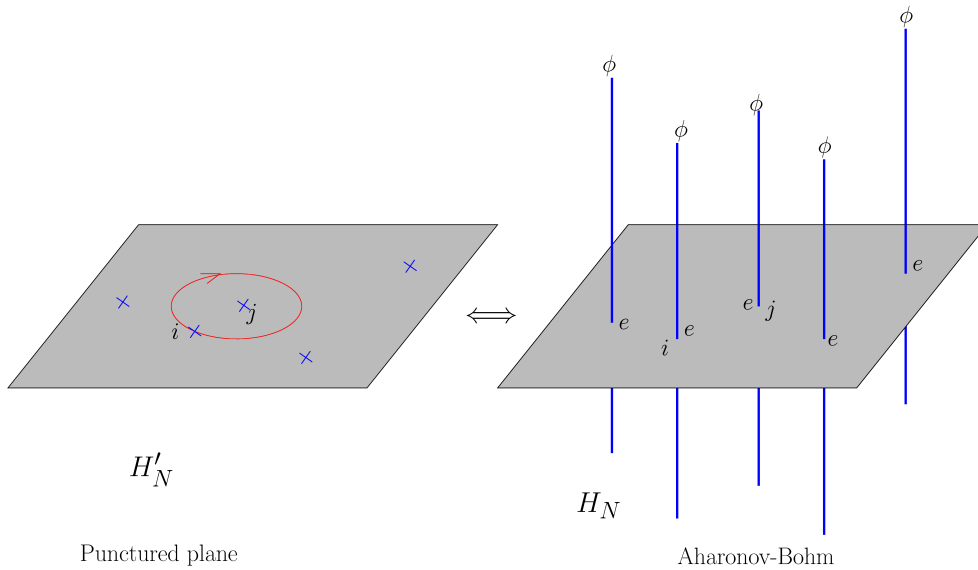

Figure 2: The two equivalent formulations of anyon statistics in terms, on the left, of a punctured plane and, on the right, of usual bosonic particles interacting via topological A-B interactions. The loop of particle i around particle j cannot be continuously deformed to nothing due to the topological obstruction materialized by the puncture at the location of particle j.

Computing the field strength one obtains

$$\frac{\alpha}{e} \vec{\partial}_i \wedge \sum_{j, j \neq i} \frac{\vec{k} \wedge \vec{r}_{ij}}{r_{ij}^2} = \frac{2\pi\alpha}{e} \sum_{j, j \neq i} \delta(\vec{r}_{ij}) \tag{10}$$

meaning that each particle carries an infinite singular magnetic field with flux $\phi = 2\pi\alpha/e$. The gauge transformation is singular since it does not preserve the field strength (which vanishes in the multivalued gauge and is singular in the regular gauge). This is due to the singular behavior of the gauge parameter θ_{ij} when particle i come close to particle j, thus the singular Dirac $\delta(\vec{r}_{ij})$ function in the field strength.

It is not surprising that topological A-B interactions are at the heart of quantum statistics. In its original form, the A-B effect [6] consists in the phase shift in electron interference due to the electromagnetic field, determined by the phase factor $\exp[(ie/\hbar c)\int_\gamma A_\mu dx^\mu]$ along a closed curve γ passing through the beam along which the field strength vanishes. This effect[1] is counter-intuitive to the usual understanding that the influence of a classical electromagnetic field on a charged particle can only occur through the local action of the field strength. In the context of quantum statistics, it means that non trivial statistics arise through topological "infinite"-distance interactions where no classical forces are present, as it should and as it is the case for Bose and Fermi statistics. Finally, singular magnetic fields give an orientation to the plane, which, as already said, shows up in $\sigma_i \neq \sigma_i^{-1}$.

All this can be equivalently restated in a Lagrange formulation which describes again the system in topological terms, i.e. free particles minimally coupled to a potential vector whose dynamics is not Coulomb-like (Maxwell Lagrangian) but rather Chern-Simons [9]

$$L_N = \sum_{i=1}^{N}(\frac{1}{2}m\vec{v}_i^2 + e(\vec{A}(\vec{r}_i)\vec{v}_i - A^0(\vec{r}_i))) - \frac{\kappa}{2}\epsilon_{\mu\nu\rho}\int d^2\vec{r}A^\mu\partial^\nu A^\rho \tag{11}$$

with $\epsilon_{\mu\nu\rho}$ the completely antisymmetric tensor (the metric is $(+, -, -)$, $x^\mu = (t, \vec{r}) = (t, x, y), A_\mu = (A_o, A_x, A_y), \epsilon_{012} = \epsilon^{012} = +1$). Solving the Euler-Lagrange equations, in particular

$$\partial^\sigma \frac{\delta L_N}{\delta(\partial^\sigma A_0)} = \frac{\delta L_N}{\delta A^0} \rightarrow \kappa\vec{\partial}\wedge\vec{A}(\vec{r}) = e\sum_{j=1}^{N}\delta(\vec{r} - \vec{r}_j) \tag{12}$$

leads to a magnetic field proportional to the density of particles in accordance with (10). Solving this last equation for $\vec{A}(\vec{r})$ in the Coulomb gauge gives

$$\vec{A}(\vec{r}) = \frac{e}{2\pi\kappa}\sum_{j=1}^{N}\frac{\vec{k}\wedge(\vec{r} - \vec{r}_j)}{(\vec{r} - \vec{r}_j)^2} \tag{13}$$

in accordance with the A-B potential vector (9). Here again there is no Lorentz force, the potential vector is a pure gauge, the Chern-Simons term is metric independent, and the field strength is directly related to the matter current.

Coming back to the Hamiltonian formulation (8), one might ask how the exclusion of the diagonal of the configuration space materializes in the Hamiltonian formulation. One way to look at it is perturbation theory [10, 11]. Let us simplify the problem by considering the standard A-B problem, i.e. a charged particle in the plane coupled to a flux tube at the origin with the Hamiltonian

$$H = \frac{1}{2m}(\vec{p} - \alpha\frac{\vec{k}\wedge\vec{r}}{r^2})^2 \tag{14}$$

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[1]The effect was first experimentally confirmed by R. G. Chambers [7], then by A. Tonomura [8].

Let us see what happens close to Bose statistics when $\alpha \simeq 0$ (by periodicity α can always been chosen in $[-1/2, +1/2]$, an interval of length 1, since in the one-body case one quantum of flux can always be gauged away via a regular gauge transformation). The A-B spectrum [6] is given by the Bessel functions

$$\psi(\vec{r}) = e^{il\theta} J_{|l-\alpha|}(kr) \qquad E = \frac{k^2}{2m} \qquad (15)$$

with wavefunctions vanishing close to the origin $r \to 0$ as $J_{|l-\alpha|}(kr) \simeq r^{|l-\alpha|}$. When the angular momentum $l \neq 0$, this is the only possible locally square-integrable function. However, when $l = 0$, one could have as well $J_{-|\alpha|}(kr)$ as a solution, since it is still locally square-integrable even though it diverges at the origin as $r^{-|\alpha|}$. In principle, the general solution in the $l = 0$ sector should be a linear combination of $J_{|\alpha|}(kr)$ and $J_{-|\alpha|}(kr)$, introducing an additional scale in the coefficient of the linear combination [12]. Restricting the space of solutions as in (15), i.e. wavefunctions vanishing at the origin, means that a short-range repulsive prescription has been imposed on the behavior of the wavefunctions when the particle comes close to the flux tube. One can give a more precise formulation of this fact by trying to compute in perturbation theory the spectrum (15). Expanding the square in the Hamiltonian (14), one finds that the α^2/r^2 term, which is as singular as the kinetic term, is divergent at second order in perturbation theory in the $l = 0$ sector. It follows that perturbation theory is not well defined in the problem as defined by the Hamiltonian (14). A renormalization has to be implemented: one realizes that by adding the counterterm $\pi|\alpha|\delta(\vec{r})$ to (14), i.e. by considering

$$H = \frac{1}{2m}(\vec{p} - \alpha \frac{\vec{k} \wedge \vec{r}}{r^2})^2 + \frac{2\pi|\alpha|}{m}\delta(\vec{r}) \qquad (16)$$

the perturbative divergences due to the α^2/r^2 term are exactly cancelled by those arising from the $\pi|\alpha|\delta(\vec{r})$ term at all orders in perturbation theory, giving back the spectrum (15). Physically, this repulsive δ contact term means that the particle is prevented from penetrating the core of the flux tube where the field strength is infinite, thus the (at least) $r^{|\alpha|}$ behavior when $r \to 0$. Note that this has been achieved without introducing any additional scale in the problem.

Clearly, in the N-body A-B anyon formulation of the model, the corresponding renormalized Hamiltonian should read

$$H_N = \sum_{i=1}^{N} \frac{1}{2m}(\vec{p_i} - \alpha \sum_{j \neq i} \frac{\vec{k} \wedge \vec{r}_{ij}}{r_{ij}^2})^2 + \frac{2\pi|\alpha|}{m} \sum_{i \neq j} \delta(\vec{r}_{ij}) \qquad (17)$$

realizing the quantum mechanical exclusion of the diagonal of the configuration space in terms of contact repulsive interaction between particles. Note that the term $\pi|\alpha| \sum_{i \neq j} \delta(\vec{r}_{ij})$ in (17) can also be viewed [11] as the Pauli spin coupling of the spin of the particles to the singular magnetic field (10) associated to the flux tubes.

The anyon model defined in (17) is properly defined as far as short-distance considerations are concerned. It is the interacting formulation for regular wavefunctions of the free particles formulation for multivalued wavefunctions. Both Hamiltonians H_N and H'_N are equivalent, the former being more familiar in terms of usual quantum mechanics, the latter more relevant to study braiding and winding properties.

The anyon model has been the subject of numerous studies in the eighties and the nineties [13], some of them analytical, starting with the 2-body case which is solvable since its relative 2-body problem is the usual A-B problem (14) with an even (Bose) angular momentum l. The exact solution [1, 5] for the relative 2-body problem is given by (15), l being an even integer, therefore when α is odd, $l - \alpha$ is odd, corresponding to Fermi statistics (the periodicity $\alpha \to \alpha \pm 2p$ is manifest in the shift $l \to l - \alpha$). These studies were followed by the 3-body [14] and then the N-body problem [15]. Statistical mechanics was also considered (second virial coefficient [16, 17], third virial coefficient [18]). However, it soon became apparent that a complete N-body spectrum was out of reach, to the exception of particular classes of exact eigenstates generalizing the 2-body eigenstates. Numerical [19] as well as semi-classical [20] studies were performed giving indications on the low energy N-body spectrum. A systematic study of the model was achieved at first [21] and at second order [22] in perturbation theory (at second order the complexity of the model shows up clearly). Numerical studies [23], taking some input from the perturbative results, were performed for the 3rd and 4th virial coefficients. Last but not least, on the experimental side, Laughlin quasiparticles [24] were put forward as the elementary excitations of highly-correlated fractional quantum Hall electron fluids [25]. They were supposed to carry a fractional charge and to obey anyon statistics [26], a fact confirmed by Berry phase calculations, at least for quasiholes [27] (for quasiparticles the situation is less clear). The quasiparticles can propagate quantum-coherently in chiral edge channels, and constructively or destructively interfere. Unlike electrons, the interference condition for Laughlin quasiparticles has a non-vanishing statistical contribution which might be observed experimentally [28].

Some kind of simplification had to be made to render the model more tractable, and possibly solvable, at least in a certain sector. One realized that this was the case if one considered, in addition to the singular statistical magnetic field, an external homogeneous magnetic field perpendicular to the plane, to which the charge of the anyons couple. In the case of a strong magnetic field, by projecting the system of anyons coupled to the magnetic field in its LLL, the model becomes solvable meaning that one can find a class of N-body eigenstates which interpolates continuously from the LLL-Bose to the LLL-Fermi eigenstates basis: this is the LLL-anyon model [29].

2   The LLL-anyon model

From now on, let us set the mass of the particles $m = 1$ and choose the statistical parameter $\alpha \in [-2, 0]$. It is understood that all the results below are obtained for α in this interval, but they can be periodically continued to the whole real axis. Before introducing an external magnetic field, let us come back to the anyon Hamiltonian (17) and take advantage of wavefunctions vanishing at least as $r_{ij}^{-\alpha}$ when $r_{ij} \to 0$ (exclusion of the diagonal of the configuration space in the quantum mechanical formulation) by encoding this short distance behavior in the N-body bosonic wave function [10]

$$\psi(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) = \prod_{i<j} r_{ij}^{-\alpha} \tilde{\psi}(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N) \tag{18}$$

$\tilde{\psi}(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$ is regular but does not have to vanish at coinciding points. From H_N in (17) one can compute the new Hamiltonian $\tilde{H}_N$ acting on

$\tilde{\psi}(\vec{r}_1, \vec{r}_2, ..., \vec{r}_N)$. Since H_N is itself obtained from the free Hamiltonian H'_N in (5) via the singular gauge transformation (6), it is more transparent to start directly from the free formulation. In complex notation (the free Hamiltonian is $H'_N = -2\sum_{i=1}^{N} \partial_i \bar{\partial}_i$) the wavefunction redefinitions (6) and (18) combined take the simple form

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{-\alpha} \tilde{\psi}(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) \qquad (19)$$

The Jastrow-like prefactor $\prod_{i<j} z_{ij}^{-\alpha}$ in (19) encodes in the wavefunction the essence of anyon statistics: topological braiding phase and short-distance contact exclusion behavior. It is immediate that $\tilde{H}_N$ rewrites as

$$\tilde{H}_N = -2\sum_{i=1}^{N} \partial_i \bar{\partial}_i + 2\alpha \sum_{i<j} \frac{1}{z_i - z_j}(\bar{\partial}_i - \bar{\partial}_j) \qquad (20)$$

It is a non-Hermitian Hamiltonian (the transformation (19) is non-unitary), but it has a simple form, linear in α and well defined in perturbation theory (it is perturbatively divergence free). Any analytic wavefunction of the z_i's is a N-body eigenstate of $\tilde{H}_N$, and therefore of the N-anyon Hamiltonian (17) taking into account (18). Analytical eigenstates are known to live in the LLL of a magnetic field, if such a field were present. Let us couple the electric charge of each anyon to an external magnetic field B perpendicular to the plane such that by convention $eB > 0$ and let us denote by $\omega_c = eB/2$ half its cyclotron frequency. One now starts from the Landau Hamiltonian

$$H'_N = -2\sum_i (\partial_i - \frac{\omega_c}{2}\bar{z}_i)(\bar{\partial}_i + \frac{\omega_c}{2}z_i) \qquad (21)$$

In a magnetic field, the 1-body eigenstates have a long-distance Landau exponential behavior $\exp(-\frac{1}{2}\omega_c z_i \bar{z}_i)$. Let us also encode this behavior in the wavefunction redefinition (19) so that it becomes

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) =$$
$$\prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i \bar{z}_i) \tilde{\psi}(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) \qquad (22)$$

One obtains

$$\tilde{H}_N = -2\sum_{i=1}^{N} (\partial_i \bar{\partial}_i - \omega_c \bar{z}_i \bar{\partial}_i) + 2\alpha \sum_{i<j} \frac{1}{z_i - z_j}(\bar{\partial}_i - \bar{\partial}_j) + N\omega_c \qquad (23)$$

where the trivial constant energy shift from the Pauli coupling to the magnetic field has been ignored. As announced, $\tilde{H}_N$ acts trivially on N-body eigenstates made of symmetrized products of analytic 1-body LLL eigenstates

$$\sqrt{\frac{\omega_c^{l_i+1}}{\pi l_i!}} z_i^{l_i}; \quad l_i \geq 0; \quad E = \omega_c \qquad (24)$$

(in (24) the Landau exponential term is missing since it has already been taken into account in (22)). So, up to an overall normalization,

$$\tilde{\psi}(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \text{Sym} \prod_{i=1}^{N} z_i^{l_i}; \quad 0 \le l_1 \le l_2 \le ... \le l_N \qquad (25)$$

is an eigenstate with a degenerate N-body energy, $E_N = N\omega_c$, a mere reflection of the fact that there are N particles in the LLL. From (22) and (25) one finally gets

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i \bar{z}_i) \text{Sym} \prod_{i=1}^{N} z_i^{l_i};$$
$$0 \le l_1 \le l_2 \le ... \le l_N \quad (26)$$

The basis (26) continuously interpolates when $\alpha = 0 \to -1$ from the complete LLL-Bose N-body basis to the complete LLL-Fermi N-body basis. Indeed, when $\alpha = -1$,

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i \bar{z}_i) \prod_{i<j} z_{ij} \text{Sym} \prod_{i=1}^{N} z_i^{l_i};$$
$$0 \le l_1 \le l_2 \le ... \le l_N \quad (27)$$

is equivalent to

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i \bar{z}_i) \text{Antisym} \prod_{i=1}^{N} z_i^{l'_i};$$
$$0 < l'_1 < l'_2 < ... < l'_N \quad (28)$$

i.e. the LLL fermionic basis. One has therefore obtained a complete LLL-Bose $\to$ LLL-Fermi interpolating basis which allows, in principle, for a complete knowledge of the LLL-anyon system with statistics intermediate between Bose and Fermi statistics.

One could ask about going beyond the Fermi point $\alpha = -1$ up to the Bose point $\alpha = -2$. This question is related to the validity of the LLL projection, since ignoring higher Landau levels amounts to assuming that excited non LLL states above the N-body LLL ground state have a non vanishing gap. Considerations around the Fermi point, as well as numerical and semiclassical analysis, support [29] this scheme as long as α does not come close to -2. However, when $\alpha \to -2$, known linear as well as unknown nonlinear non LLL eigenstates do join the LLL ground state [31]. Said differently, the LLL-anyon basis (26) does not constitute a complete LLL-Bose basis when $\alpha \to -2$, i.e. some N-body LLL bosonic quantum numbers are missing at this point. We will come back to this issue later.

One has not seen yet any α dependence in the N-body energy, a situation already encountered in the 1-body A-B problem, where the free continuous energy spectrum (15) is α-independent. This is due to the fact that a magnetic field does not confine particles: classical orbits are circular cyclotron orbits, but their centers, due to translation invariance, are located anywhere in the plane. Translation invariance
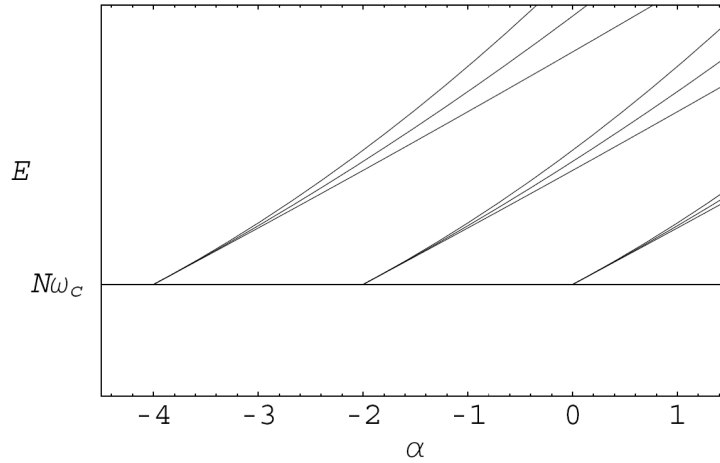
Figure 3: Linear and non linear non LLL eigenstates merge in the LLL ground state at the bosonic values of α.

in turn gives, in quantum mechanics, a Landau spectrum which is l_i independent, and therefore infinitely degenerate[2]. The degeneracy factor scales as the infinite surface V of the 2d sample: it is the flux of the magnetic field counted in units of the flux quantum $\phi_0 = 2\pi/e$ (in units $\hbar = 1$)

$$N_L = \frac{VB}{\phi_0} \tag{29}$$

Statistical interactions being topological interactions, one does not expect, in the infinite plane limit, any effect on the N-body energies. To see such an effect, one has to introduce a long-distance confinement, like putting the particles in a box. Let us rather introduce [30] a more convenient harmonic well confinement where the particles are trapped, so that the Landau Hamiltonian (21) becomes

$$H'_N = -2\sum_{i=1}^{N}(\partial_i - \frac{eB}{4}\bar{z}_i)(\bar{\partial}_i + \frac{eB}{4}z_i) + \frac{1}{2}\omega^2\sum_{i=1}^{N}z_i\bar{z}_i \tag{30}$$

The virtue of the harmonic confinement is to lift the degeneracy with respect to the angular momentum l_i of the 1-body Landau eigenstates: the harmonic LLL spectrum[3] becomes

$$\sqrt{\frac{\omega_t^{l_i+1}}{\pi l_i!}}z_i^{l_i}\exp(-\frac{1}{2}\omega_t z_i\bar{z}_i); \quad l_i \geq 0; \quad E = (\omega_t - \omega_c)(l_i + 1) + \omega_c \tag{32}$$

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[2]From this point of view one can argue that the Landau spectrum is continuous, albeit being made of discrete Landau levels, due to the infinite degeneracy on each level.

[3]The complete 2d harmonic Landau spectrum is, with the convention $eB > 0$,

$$\omega_t(2n_i + l_i + 1) - l_i\omega_c; \quad n_i \geq 0, \quad l_i \in Z \tag{31}$$

The LLL quantum numbers are $n_i = 0$ and $l_i \geq 0$.

where $\omega_t = \sqrt{\omega^2 + \omega_c^2}$. Each harmonic LLL level in (32) has now a finite degeneracy, with an eigenstate still analytic in z_i, up to the long-distance harmonic Landau combined exponential behavior. Let us take into account this exponential behavior in the redefinition of the free N-body wavefunction so that (22) now becomes

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) =$$
$$\prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{1}{2}\omega_t \sum_{i=1}^{N} z_i\bar{z}_i)\tilde{\psi}(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) \quad (33)$$

Starting from the Hamiltonian (30) one obtains [32, 33]

$$\tilde{H}_N = -2\sum_{i=1}^{N}(\partial_i\bar{\partial}_i - \frac{\omega_t + \omega_c}{2}\bar{z}_i\bar{\partial}_i - \frac{\omega_t - \omega_c}{2}z_i\partial_i)$$
$$+ 2\alpha\sum_{i<j}\Big[\frac{1}{z_i - z_j}(\bar{\partial}_i - \bar{\partial}_j) - \frac{\omega_t - \omega_c}{2}\Big] + N\omega_c \quad (34)$$

Again let us act on N-body eigenstates made, in analogy with (25), of symmetrized products of the 1-body harmonic LLL eigenstates (32)

$$\tilde{\psi}(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \text{Sym} \prod_{i=1}^{N} z_i^{l_i}; \quad 0 \leq l_1 \leq l_2 \leq ... \leq l_N \quad (35)$$

Acting on this basis, the Hamiltonian (34) rewrites as

$$\tilde{H}_N = (\omega_t - \omega_c)\Big[\sum_{i=1}^{N} z_i\partial_i - \alpha\frac{N(N-1)}{2} + N\Big] + N\omega_c \quad (36)$$

so that the N-anyon energy spectrum is

$$E_N = (\omega_t - \omega_c)\Big[\sum_{i=1}^{N} l_i - \alpha\frac{N(N-1)}{2} + N\Big] + N\omega_c \quad (37)$$

The N-anyon spectrum (37) is a sum of 1-body harmonic LLL spectra shifted by the 2-body statistical term $-(\omega_t - \omega_c)\alpha N(N-1)/2$. The effect of the harmonic well has been not only to lift the degeneracy with respect to the l_i's, but also to make the energy dependence on α explicit. When computing thermodynamical quantities like the equation of state, the harmonic well regulator will also be needed to compute finite quantities in a finite "harmonic" box, and then take the thermodynamic limit, by letting $\omega \to 0$ in an appropriate way.

The resulting eigenstates from (33)

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{1}{2}\omega_t \sum_{i=1}^{N} z_i\bar{z}_i)\text{Sym} \prod_{i=1}^{N} z_i^{l_i};$$
$$0 \leq l_1 \leq l_2 \leq ... \leq l_N \quad (38)$$

are called "linear states" since their energy (37) varies linearly with α. As already stressed, they constitute a set of exact N-body eigenstates which is only a small part of the complete N-body spectrum, which remains mostly unknown. However, what makes, in the LLL context, these linear states particularly interesting is that they continuously interpolate when $\alpha = 0 \to -1$ from the complete harmonic LLL-Bose basis to the complete harmonic LLL-Fermi basis.

Before turning to LLL-anyon thermodynamics, let us reconsider the physical charge-flux composite interpretation of the anyon model, where the charges are now coupled to an external magnetic field. A given particle, say the Nth, sees a "positive" ($eB > 0$) magnetic field perpendicular to the plane, and $N-1$ "negative" ($e\phi = 2\pi\alpha < 0$, $\alpha \in [-1, 0]$) point vortices piercing the plane at the positions of the other particles. This is a screening regime: in the large N limit where a mean field picture is expected to be valid, the more α is close to the fermionic point $\alpha = -1$, the more the external magnetic field is screened by the mean magnetic field associated with the vortices. In terms of the total (external + mean) magnetic field $\langle B \rangle$ that the Nth particle sees, or rather in terms of its flux $V\langle B \rangle$, or, when counted in units of the flux quantum, in terms of the Landau degeneracy $\langle N_L \rangle$, one has

$$V\langle B\rangle/\phi_0 = (VB)/\phi_0 + (N-1)\phi/\phi_0 \quad \text{i.e.} \quad \langle N_L \rangle = N_L + (N-1)\alpha \qquad (39)$$

Moving away from the Bose point, i.e. $\alpha \leq 0$, as N increases the number $\langle N_L \rangle$ of 1-body quantum states available for the Nth particle in the LLL of $\langle B \rangle$ decreases. This sounds reasonable, bearing in mind that a fermion occupies a quantum state to the exclusion of others (Pauli exclusion), whereas bosons can condense (Bose condensation). Introducing the LLL filling factor

$$\nu = \frac{N}{N_L} \qquad (40)$$

one deduces from (39) a maximal critical filling [29] for which the screening is total, $\langle N_L \rangle = 0$

$$\nu = -\frac{1}{\alpha} \qquad (41)$$

This is nothing but recognizing once more that bosons ($\alpha = 0$) can infinitely fill a quantum state ($\nu = \infty$), whereas fermions ($\alpha = -1$) are at most one per quantum state ($\nu = 1$). In between, one finds that there are at most $-1/\alpha$ anyons per quantum state.

Interestingly enough, Haldane/exclusion statistics definition[4] happens to coincide with (39): for a gas of particles obeying Haldane/ exclusion statistics [34] with statistical parameter $g \in [0, 1]$, given N_L degenerate energy levels and $N - 1$ particles already populating the levels, the number d_N of quantum states still available for the Nth particle is given by (39) where $-\alpha$ is replaced by g

$$d_N = N_L - (N-1)g \qquad (42)$$

On the one hand, Haldane's definition (42) stems from an arbitrary combinatorial point of view, inspired by the Bose and Fermi counting of states. On the other hand, in the LLL-anyon model, (39) is obtained from a somehow ad-hoc mean field ansatz. We will come back to these issues in the next section.

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[4]This is Haldane's statistics for one particle species. It can be generalized to the multispecies case.

3 LLL-anyon thermodynamics

Let us rewrite the N-body energy (37) as [37]

$$E_N = \sum_{i=1}^{N}(\epsilon_0 + l_i\tilde{\omega}) - \alpha\frac{N(N-1)}{2}\tilde{\omega}; \quad 0 \le l_1 \le l_2 \le ... \le l_N \tag{43}$$

with $\tilde{\omega} = (\omega_t - \omega_c)$ and $\epsilon_0 = \omega_c$. Introducing the fugacity z and the inverse temperature β, one wants to compute the thermodynamic potential

$$\ln Z(\beta, z) = \ln(\sum_{N=0}^{\infty} z^N Z_N); \quad Z_0 = 1 \tag{44}$$

where $Z(\beta, z)$ is the grand partition function defined in terms of the N-body partition functions $Z_N = \text{Tr}\exp(-\beta H'_N) = \text{Tr}\exp(-\beta H_N) = \text{Tr}\exp(-\beta\tilde{H}_N)$. The thermodynamic potential rewrites as $\ln Z(\beta, z) = \sum_{n=1}^{\infty} b_n z^n$ where, at order z^n, the cluster coefficient b_n only requires the knowledge of the Z_i's, with $i \le n$. One is interested in evaluating the thermodynamic potential in the thermodynamical limit, i.e. ω is small, which means, here, that the dimensionless quantity $\beta\omega$ is small. The N-body spectrum, as given in (43), allows to compute, at leading order in $\beta\omega \to 0$, the Z_i's for $i \le n$, and thus the b_n's

$$b_n = \frac{1}{\beta\tilde{\omega}}\frac{e^{-n\beta\omega_c}}{n^2}\prod_{k=1}^{n-1}\frac{k + n\alpha}{k}; \quad b_1 = \frac{1}{\beta\tilde{\omega}}e^{-\beta\omega_c} \tag{45}$$

One has still to give a meaning, in the thermodynamic limit $\beta\omega = 0$, to the scaling factor $1/(\beta\tilde{\omega})$ in (45). To this purpose, one temporarily switches off the anyonic interaction and the external magnetic field, and considers a quantum gas of non interacting harmonic oscillators per se. One asks, when $\beta\omega \to 0$, for its cluster coefficients to yield the infinite box (plane wave) cluster coefficients. At order n in the cluster expansion, in d dimensions, one obtains [10]

$$\lim_{\beta\omega\to0}\left(\frac{1}{n(\beta\omega)^2}\right)^{\frac{d}{2}} = \frac{V}{\lambda^d} \tag{46}$$

where $\lambda = \sqrt{2\pi\beta}$ is the thermal wavelength and V is the d-dimensional infinite volume (in $d = 2$ dimensions, V is, as defined above, the infinite area of the 2d sample). Using the thermodynamic limit prescription (46), the cluster coefficient (45) rewrites, in the thermodynamic limit, as [29]

$$b_n = N_L\frac{e^{-n\beta\omega_c}}{n}\prod_{k=1}^{n-1}\frac{k + n\alpha}{k}; \quad b_1 = N_L e^{-\beta\omega_c} \tag{47}$$

The cluster expansion $\ln Z(\beta, z) = \sum_{n=1}^{\infty} b_n z^n$, as a power series of $ze^{-\beta\omega_c} < 1$, can be summed up

$$\ln Z(\beta, z) = N_L \ln y(ze^{-\beta\omega_c}) \tag{48}$$

where $y(ze^{-\beta\omega_c})$, a function of the variable $ze^{-\beta\omega_c}$, is such that

$$\ln y = ze^{-\beta\omega_c} + \sum_{n=2}^{\infty}\frac{(ze^{-\beta\omega_c})^n}{n}\prod_{k=1}^{n-1}\frac{k + n\alpha}{k} \tag{49}$$

It obeys [29]

$$y - ze^{-\beta\omega_c}y^{1+\alpha} = 1 \tag{50}$$

and has in turn a power series expansion [38]

$$y = 1 + ze^{-\beta\omega_c} + \sum_{n=2}^{\infty} (ze^{-\beta\omega_c})^n \prod_{k=2}^{n} \frac{k+n\alpha}{k} \tag{51}$$

From (48) one infers that $Z(\beta, z) = y^{N_L}$ so that [32, 39]

$$Z(\beta, z) = y^{N_L} = 1 + N_L ze^{-\beta\omega_c} + N_L \sum_{N=2}^{\infty} (ze^{-\beta\omega_c})^N \prod_{k=2}^{N} \frac{k+N_L+N\alpha-1}{k} \tag{52}$$

Clearly, from (52), the N-body partition function Z_N is

$$Z_N = N_L e^{-N\beta\omega_c} \prod_{k=2}^{N} \frac{k+N_L+N\alpha-1}{k} \tag{53}$$

It is, by construction, positive. Necessarily, α and N_L being given, N has to be such that $N_L + N\alpha \geq 0$. This always is the case as long as N is finite, since N_L scales like the infinite surface of the 2d sample. In the thermodynamic limit, where $N \to \infty$, the condition $N_L + N\alpha \geq 0$ implies for the filling factor

$$\nu \leq -\frac{1}{\alpha} \tag{54}$$

It is rather striking that the RHS of (54), which has just been derived from the exact computation of the cluster coefficients from the N-body spectrum, is nothing but the critical filling (41) obtained in the mean field approach when the screening is total.

The "degeneracy" associated with N anyons populating the LLL quantum states is, from (53),

$$N_L \prod_{k=2}^{N} \frac{k+N_L+N\alpha-1}{k} = \frac{N_L}{N!} \frac{(N+N_L+N\alpha-1)!}{(N_L+N\alpha)!} \tag{55}$$

where a factorial with a negative argument has to be understood as $(-p)! = \lim_{x\to 0} (-p+x)!$.

When $\alpha = 0$, this is the usual Bose counting factor for the number of ways to put N bosons in N_L states

$$\frac{(N+N_L-1)!}{N!(N_L-1)!} \tag{56}$$

When $\alpha = -1$, this is the Fermi counting factor $N_L!/(N!(N_L - N)!)$. If one considers for a moment the statistical parameter to be a negative integer $\alpha \leq -1$, the degeneracy (55) still allows for a combinatorial interpretation [38] : provided again that $N_L + N\alpha \geq 0$, it is the number of ways to put N particles on a circle consisting of N_L quantum states such that there are at least $-\alpha-1$ empty states in between two occupied states. When $\alpha = -1$, this is nothing but the usual exclusion mechanism for fermions (one fermion at most per quantum state). When $\alpha \leq -1$, i.e. beyond

the Fermi point, more and more states are excluded between two filled states. In the case of interest α in $[-1, 0]$, one has a "fractional" exclusion where one can put more than one particle per quantum state according to the fractional α, but not infinitely many as in the Bose case.

The degeneracy (55) originates from the exact N-body spectrum (37). In the case of Haldane statistics as defined in (42), there is no Hamiltonian and no N-body spectrum to begin with. One rather starts from the Bose counting factor (56) and bluntly replaces, in accordance with (42), N_L by $N_L - (N-1)g$ to obtain

$$\frac{(N_L - (N-1)(g-1))!}{N!(N_L - (N-1)g - 1)!}; \tag{57}$$

which indeed interpolates, when $g = 1$, to the Fermi counting factor. The degeneracy (57) is similar to (55): if one allows the exclusion parameter g to be an integer, it counts [38] the number of ways to put N particles on a line of finite length consisting of N_L quantum states such that there are at least $g - 1$ empty states in between two occupied states. Up to boundary conditions on the space of available quantum states (periodic versus infinite wall), both counting (55, 57) are identical. In the thermodynamic limit when N becomes large, boundary conditions should not play a role anymore: not surprisingly, starting from (57) and following the usual route of statistical mechanics [40] (saddle-point approximation) leads, in the thermodynamic limit, to the same LLL-anyon thermodynamic potential given by the equations (48) and (50), where the anyonic parameter $-\alpha$ is replaced by the exclusion parameter g.

Note that the grand partition factorization $Z(\beta, z) = y^{N_L}$ in (48) could suggest [41] an interpretation of y as a LLL-anyon grand-partition function for a single quantum state at energy ω_c, on the same footing as, when $\alpha = 0$ or $\alpha = -1$, $y = (1 \mp ze^{-\beta\omega_c})^{\mp 1}$ is indeed the single quantum state grand partition function for a Bose or Fermi gas. This interpretation is not possible for the reason advocated above: it would yield, as soon as α is fractional, negative N-body partition functions. This is clearly impossible: the N-body anyonic system is, except in the Bose and Fermi cases, truly interacting and therefore its statistical mechanics is by no means factorisable to a single-state statistical mechanics.

From (48, 50), the average energy $\bar{E} \equiv -\partial \ln Z(\beta, z)/\partial\beta$ and the average particle number $\bar{N} \equiv z\partial \ln Z(\beta, z)/\partial z$ or, equivalently, the filling factor $\nu = \bar{N}/N_L$, can be computed. ν satisfies

$$y = 1 + \frac{\nu}{1 + \alpha\nu} \tag{58}$$

or, equivalently, using (50)

$$ze^{-\beta\omega_c} = \frac{\nu}{(1 + (1+\alpha)\nu)^{1+\alpha}(1 + \alpha\nu)^{-\alpha}} \tag{59}$$

When $\alpha \neq 0$ and $\alpha \neq -1$, this equation cannot in general be solved analytically, except in special cases like $\alpha = -1/2$ (semions). The equation of state follows

$$\beta PV = \ln(1 + \frac{\nu}{1 + \alpha\nu}) \tag{60}$$

In all these equations, it is understood from (54) that $\nu \leq -1/\alpha$. When $\nu = -1/\alpha$, the pressure diverges, a manifestation of the fact that there are as many anyons

as possible in the LLL, higher Landau levels being forbidden by construction. One also notes that, for the degenerate LLL gas, the filling factor in (59) is nothing but the mean occupation number n at energy $\epsilon = \omega_c$ and fugacity z. As expected, (59) at $\alpha = 0$ gives the standard Bose mean occupation number $n = ze^{-\beta\epsilon}/(1 - ze^{-\beta\epsilon})$, whereas at $\alpha = -1$ it gives the Fermi mean occupation number $n = ze^{-\beta\epsilon}/(1+ze^{-\beta\epsilon})$.

The entropy $S \equiv \ln Z(\beta, z) + \beta\bar{E} - (\ln z)\bar{N}$ is (trivially $\bar{E} = \bar{N}\omega_c$ since the N particles are in the LLL)

$$S = N_L\big[(1 + \nu(1 + \alpha))\ln(1 + \nu(1 + \alpha)) - (1 + \nu\alpha)\ln(1 + \nu\alpha) - \nu\ln\nu\big] \qquad (61)$$

It vanishes when $\nu = -1/\alpha$, an indication that the N-body LLL anyon eigenstate is not degenerate at the critical filling. From (37), one infers that the N-body eigenstate of lowest energy has all its one-body orbital momenta quantum numbers $l_i = 0$. It follows from (26) that, in the thermodynamic limit at the critical filling, the LLL-anyon non-degenerate groundstate wavefunction is

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{-\alpha} \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i\bar{z}_i); \quad \nu = -\frac{1}{\alpha} \qquad (62)$$

with total angular momentum

$$L = \frac{N(N-1)}{2\nu} \qquad (63)$$

The pattern in (62) is reminiscent of the Laughlin wavefunctions at FQHE fillings $\nu = 1/(2m+1)$

$$\psi(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{2m+1} \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i\bar{z}_i);$$

$$\nu = \frac{1}{2m+1} \qquad (64)$$

On the one hand, Laughlin wavefunctions are fermionic, their filling factors are rational numbers smaller than 1, and they are approximate solutions to the underlying N-body Coulomb dynamics in a strong magnetic field. On the other hand, LLL-anyon wavefunctions are multivalued, their filling factor continuously interpolates between ∞ and 1, and they are exact solutions to the N-body LLL anyon problem. Still, the similarity between (62) and (64) is striking.

Trying to push (62) further beyond the Fermi point eventually up to the Bose point at $\alpha = -2$, one obtains a Bose gas at filling $\nu = 1/2$ with the non-degenerate wavefunction

$$\psi'(z_1, z_2, ..., z_N; \bar{z}_1, \bar{z}_2, ..., \bar{z}_N) = \prod_{i<j} z_{ij}^{2} \exp(-\frac{1}{2}\omega_c \sum_{i=1}^{N} z_i\bar{z}_i); \quad \nu = \frac{1}{2} \qquad (65)$$

One already knows that the LLL-anyon basis (26) is not interpolating to the complete LLL-Bose basis when $\alpha = -2$. At this point, non LLL N-body eigenstates merge in the LLL ground state to compensate for some missing bosonic quantum numbers -see Figure 3. Clearly, (65) should reproduce, by periodicity, the bosonic

non-degenerate wavefunction (62) at $\alpha = 0$, but it does not. On the same footing, when $\alpha = -2$ the critical filling should be bosonic, i.e. $\nu = \infty$, whereas $\nu = 1/2$. The unphysical critical filling discontinuity, ∞ versus $1/2$, is yet another manifestation of the missing bosonic quantum numbers. In other words, the very eigenstates which join the LLL ground state at the Bose point $\alpha = -2$ and provide for the missing quantum numbers, have the effect to smooth out the critical filling discontinuity. Still, it has been shown [35] that the stronger the magnetic field B is, the more valid (62) remains closer and closer to $\alpha = -2$. The limit $\alpha \to -2$ is, due to periodicity, the same as the limit $\alpha \to 0$ from above, which can be described as an anti-screening regime. One concludes that close to the Bose point $\alpha = 0$, the critical filling of a LLL-anyon gas is $\nu = \infty$ or $\nu = 1/2$ depending on infinitesimally moving away from the Bose point in the screening regime (the ground state wavefunction is the usual non degenerate bosonic wavefunction), or in the anti-screening regime (the ground state wavefunction is (65)). Again, the Bose point has a somehow singular behavior, a feature already encountered in perturbation theory. Note finally that the occurrence of the $\nu = 1/2$ fraction for the bosonic filling factor in the antiscreening regime is physically challenging: fast rotating Bose-Einstein condensates in the FQHE regime are expected [36] to reach a $1/2$ filling described by the Laughlin-like wavefunction (65).
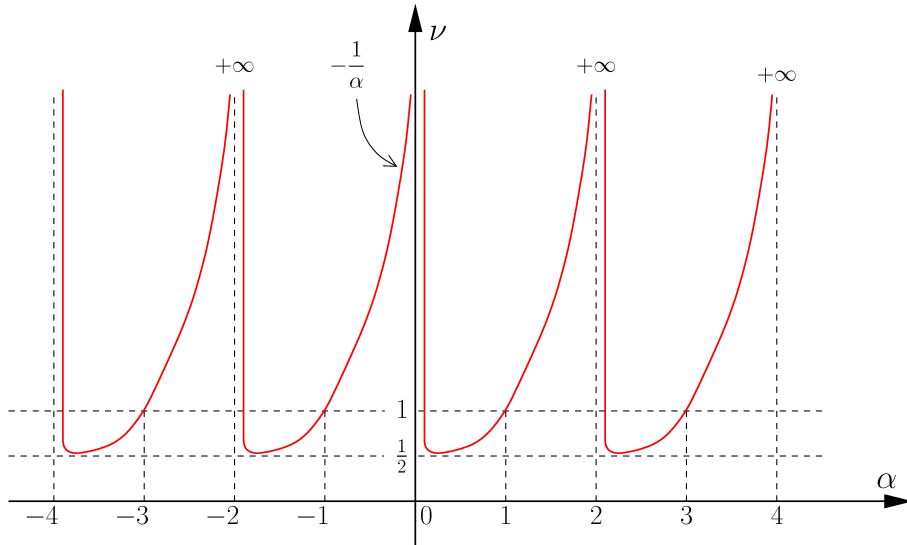

Figure 4: The critical LLL-anyon filling curve as a function of α. The critical Bose filling $\nu = \frac{1}{2}$ occurs at the Bose points in the anti-screening regime. The continuity of the critical curve at these points is restored by the non LLL eigenstates joining the LLL ground state.

So far one has been concerned with two-dimensional systems: in the thermodynamic limit, a single particle in the LLL, and, consequently, a gas of LLL-anyons, are two dimensional, as can be seen from the $N_L \simeq V$ scalings[5] of the 1-body LLL partition function $Z_{LLL} = N_L \exp(-\beta\omega_c)$ and the LLL anyon thermodynamic potential (48). Denoting by $\rho_{LLL}(\epsilon) = N_L\delta(\epsilon - \omega_c)$ the 1-body LLL density of states,

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[5]In the LLL there is only one quantum number l_i per particle, still the system is 2d.

(48) can be rewritten as

$$\ln Z(\beta, z) = \int_0^\infty \rho_{LLL}(\epsilon) \ln y(ze^{-\beta\epsilon}) d\epsilon \tag{66}$$

Convincingly, in (66) the one-body dynamics of individual particles is described by the one-body density of states, whereas the LLL anyon statistical collective behavior is encoded in the y function which depends on the statistical parameter α.

One might ask about other integrable N-body systems which would lead to the same kind of statistics. It would be tempting to define a model obeying fractional/exclusion statistics if, its one-body density of states $\rho(\epsilon)$ being given, its thermodynamic potential has the form

$$\ln Z(\beta, z) = \int_0^\infty \rho(\epsilon) \ln y(ze^{-\beta\epsilon}) d\epsilon \tag{67}$$

with

$$y - ze^{-\beta\epsilon} y^{1+\alpha} = 1 \tag{68}$$

so that

$$y = 1 + ze^{-\beta\epsilon} + \sum_{n=2}^\infty (ze^{-\beta\epsilon})^n \prod_{k=2}^n \frac{k + n\alpha}{k} \tag{69}$$

The mean occupation number follows as $n = z\partial \ln y/\partial z$. It obeys

$$y = 1 + \frac{n}{1 + \alpha n}; \quad \text{or} \quad n = \frac{y - 1}{1 - \alpha(y - 1)} \geq 0 \tag{70}$$

or, equivalently,

$$ze^{-\beta\epsilon} = \frac{n}{(1 + (1 + \alpha)n)^{1+\alpha}(1 + \alpha n)^{-\alpha}} \tag{71}$$

One has the duality relation [41]

$$1 = \frac{1}{y} + \frac{1}{\tilde{y}}; \quad \text{where} \quad \tilde{y} - (ze^{-\beta\epsilon})^{-1}\tilde{y}^{1+\frac{1}{\alpha}} = 1 \tag{72}$$

or, equivalently

$$-\alpha n - \frac{1}{\alpha}\tilde{n} = 1 \tag{73}$$

where $\tilde{n}$ is related to $\tilde{y}$ as n to y in (70). The duality relation (72,73) can be interpreted as a particle-hole symmetry relation. Setting $t = ze^{-\beta\omega_c}$, one also has a simple expression [42] for $dn(t)/dt$

$$t\frac{dn}{dt} = n(1 + (1 + \alpha)n)(1 + \alpha n) \tag{74}$$

All these equations have been understood as arising microscopically from the LLL anyon Hamiltonian with one-body density of states $\rho(\epsilon) = \rho_{LLL}(\epsilon)$. It happens that it is possible to find another N-body microscopic Hamiltonian which leads to the thermodynamics (67). Consider, in one dimension, the integrable N-body Calogero model [43] with inverse-square 2-body interactions

$$H_N = -\frac{1}{2}\sum_{i=}^N \frac{\partial^2}{\partial x_i^2} + \alpha(1 + \alpha)\sum_{i<j}\frac{1}{(x_i - x_j)^2} + \frac{1}{2}\omega^2\sum_{i=1}^N x_i^2 \tag{75}$$

where x_i represents the position of the i-th particle on the infinite 1d line. This model is known to describe particles with nontrivial statistics in one dimension interpolating from Bose ($\alpha = 0$) to Fermi ($\alpha = -1$) statistics. It means that the $1/x^2$ Calogero interaction is purely statistical, without any classical effect on particle motions, up to a overall reshuffling of the particles [44]. The Calogero model remains integrable when, as in (75), a confining 1d harmonic well is added. This is the harmonic Calogero model, whereas the Calogero-Sutherland model [46] would have the particles confined on a circle. The effect of the harmonic well is, as in the LLL anyon case, to lift the thermodynamic limit degeneracy in such a way that the N-body harmonic Calogero spectrum ends up depending on the Calogero coupling constant α

$$E_N = \omega\Big[\sum_{i=1}^{N} l_i - \alpha\frac{N(N-1)}{2} + \frac{N}{2}\Big]; \quad 0 \le l_1 \le l_2 \le ... \le l_N \tag{76}$$

Here the l_i's correspond to the quantum numbers of the 1-d harmonic Hermite polynomials free 1-body eigenstates

$$(\frac{\omega}{\pi})^{1/4}\frac{1}{\sqrt{2^{l_i}l_i!}}e^{-\frac{1}{2}\omega x_i^2}H_{l_i}(\sqrt{\omega}x_i); \quad l_i \ge 0; \quad E = \omega(l_i + \frac{1}{2}) \tag{77}$$

It is remarkable that (76) happens to be again of the form (43) with $\tilde{\omega} = \omega$, $\epsilon_0 = \omega/2$. Following the same steps as in the LLL-anyon case, and using again (46) while taking the thermodynamic limit $\beta\omega \to 0$, the Calogero cluster coefficients rewrite as

$$b_n = \frac{L}{\lambda}\frac{1}{n\sqrt{n}}\prod_{k=1}^{n-1}\frac{k + n\alpha}{k}; \quad b_1 = \frac{L}{\lambda} \tag{78}$$

where the infinite length of the 1d line has been denoted by L. The cluster expansion can still be resumed using (49) provided the unwanted $1/\sqrt{n}$ term in (78) is properly taken care of. Introducing the 1d plane wave momentum k

$$\frac{1}{\lambda\sqrt{n}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{-n\beta\frac{k^2}{2}} \tag{79}$$

and denoting the 1-body energy as $\epsilon = k^2/2$, one finally obtains

$$\ln Z(\beta, z) = \int_0^\infty \rho_0(\epsilon)\ln y(ze^{-\beta\epsilon})d\epsilon \tag{80}$$

where

$$\rho_0(\epsilon) = \frac{L}{\pi\sqrt{2\epsilon}} \tag{81}$$

is the free 1-body density of states in one dimension. This is not a surprise: in the thermodynamic limit $\omega \to 0$, where $l_i \to \infty$ with $l_i\omega = k_i^2/2$ kept fixed, the Hermite polynomial H_{l_i} becomes a plane wave of momentum k_i.

From (80), one concludes[6] that, in the thermodynamic limit, the Calogero model has indeed a LLL-anyon/exclusion like statistics [45] according to (67) and (68),

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[6]The same conclusion would be reached starting form the Calogero-Sutherland model and taking the corresponding thermodynamic limit, i.e. the radius of the confining circle going to infinity.

interpolating, as it should, from a free bosonic 1d gas at $\alpha = 0$ to a free fermionic 1d gas at $\alpha = -1$.

It follows that the 2d LLL-anyon and 1d Calogero models, which seem a priori unrelated, do obey the same type of statistics. This is not a coincidence. Looking at their harmonic N-body spectrum (37) and (76), one realizes that, up to an irrelevant zero-point energy, the latter is the $B \to 0$ limit of the former. This remains true in the thermodynamic limit $\omega \to 0$. So, not only (66) and (80) are of the same type, but also, when $B \to 0$, (66) has to become (80). It follows that, necessarily, the 1-body densities of states $\rho_{LLL}(\epsilon)$ and $\rho_0(\epsilon)$ satisfy $\lim_{B \to 0} \rho_{LLL}(\epsilon) = \rho_0(\epsilon)$, i.e.

$$\lim_{B \to 0} \frac{eBV}{2\pi} \delta(\epsilon - \frac{eB}{2}) = \frac{L}{\pi\sqrt{2\epsilon}} \tag{82}$$

a relation which has to be understood as arising in the thermodynamic limit $\omega \to 0$.

To arrive at (82), one could as well consider directly the 1-body harmonic LLL spectrum (32) and harmonic 1d spectrum (77)

$$E = (\omega_t - \omega_c)(l_i + 1) + \omega_c; \quad E = \omega(l_i + \frac{1}{2}) \tag{83}$$

They are such that the latter is the vanishing B limit of the former, so it is the case for the corresponding 1-body partition functions. Taking then[7] the thermodynamic limit $\beta\omega \to 0$, i.e. (46), implies the relation $\lim_{B \to 0} Z_{LLL} = Z_0$, where Z_{LLL} is, as above, the LLL partition function and Z_0 is the free partition function in one dimension. Consequently for the densities of states (the inverse Laplace transforms) the relation (82) follows. This result has its roots in the different energy gaps of the spectra (83) at small ω: in the harmonic LLL case, the gap behaves like $\omega^2/(2\omega_c)$, whereas, in the 1d harmonic case, the gap is ω.

The relation (82) could also have been understood from the 1-body eigenstates themselves. In the limit $B \to 0$, the LLL induced harmonic analytic eigenstates are, from (32),

$$\sqrt{(\frac{\omega^{l_i+1}}{\pi l_i!})} z_i^{l_i} e^{-\frac{1}{2}\omega z_i \bar{z}_i} \tag{84}$$

There is only one parameter ω left so that the states in (84) can be put in one-to-one correspondence with the Hermite polynomials (77) via the Bargmann transform

$$\sqrt{\omega^{l_i+1}} z_i^{l_i} = \omega \int_{-\infty}^{\infty} dx_i \frac{1}{\sqrt{2^{l_i}}} e^{-\omega(x_i^2 - z_i x_i \sqrt{2} + z_i^2/2)} H_{l_i}(\sqrt{\omega} x_i) \tag{85}$$

From (85) one can infer [47] that the N-body harmonic anyon eigenstates (38) are a coherent state representation of the N-body harmonic Calogero eigenstates.

From all these considerations (thermodynamics, eigenstates,...) it follows that the vanishing magnetic field limit[8] of the LLL-anyon model is the Calogero model itself. It seems paradoxical to consider such a limit in the LLL which assumes a

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[7]The order of limits is crucial here: first the limit $B \to 0$, then the thermodynamic limit $\omega \to 0$.

[8]Since one has ended up by taking the limit $B \to 0$, one could have avoided right from the beginning to introduce a B field, and started directly from the harmonic N-body anyon model. What has been done above by taking the limit $B \to 0$ is nothing but to project the harmonic anyon model on the LLL induced harmonic subspace (84) (the B field and its LLL should still be invoked to justify the selection of the LLL quantum numbers in the 2d harmonic basis) and to recognize that the projected harmonic anyon model is the harmonic Calogero model. This relation remains true in the thermodynamic limit $\omega \to 0$.

priori a strong magnetic field. Still, doing so, one has dimensionally reduced the 2d anyon model to the 1d Calogero model. This dimensional reduction has a simple geometrical interpretation. The LLL induced harmonic states (84) are localized in the vicinity of circles of radius l_i/ω. In the thermodynamic limit, one has $l_i \to \infty$ with $l_i\omega = k_i^2/2$ kept fixed. It follows that the corresponding 1d Hermite polynomials H_{l_i}, which become in this limit plane waves of momentum k_i, have a radius of localization diverging like k_i^2/ω^2. The dimensional reduction which has taken place consists in going at infinity on the edge of the plane: in the thermodynamic limit, the Calogero model can be viewed as the edge projection of the anyon model.

The LLL anyon thermodynamics, or, equivalently, the Haldane/ exclusion thermodynamics, and the Calogero thermodynamics as well, have been the subject of an intense activity since the mid-nineties. Let us mention their relevance in more abstract contexts, such as conformal field theories [48]. On the experimental side, FQHE edge currents can be modelled by quasiparticles with fractional statistics, which in turn might affect their transport properties such as the current shot noise [49, 42].

4   Minimal Difference Partitions and Trees

Up to now one has been concerned with quantum mechanical models defined by a microscopic quantum Hamiltonian. Both the LLL anyon and Calogero models have been shown to have a thermodynamics controlled by (67) and (68). Let us leave quantum mechanics and address a pure combinatorial problem, the minimal difference partition problem [50]. Consider the number $\rho(E, N)$ of partitions of an integer E into N integer parts where each part differs from the next by at least an integer p and the smallest part is $\geq l$. Usual integer partitions correspond to $p = 0$ and $l = 1$, whereas restricted partitions, where the parts have to be different, correspond to $p = 1$ and $l = 1$.
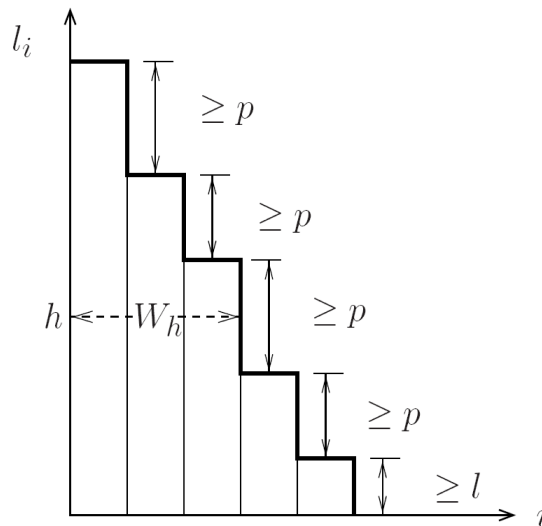

Figure 5: A minimal difference partition configuration, or Young diagram. The column heights are such that $(l_i - l_{i+1}) \geq p$ for $i = 1, 2, \ldots, N - 1$ and $l_i \geq l$. Their total height is $E = \sum_{i=1}^{N} l_i$. W_h is the width of the Young diagram at height h, i.e. the number of columns whose heights $\geq h$.

It is known that

$$\sum_E \rho(E, N)x^E = \frac{x^{lN+pN(N-1)/2}}{(1-x)(1-x^2)...(1-x^N)} \tag{86}$$

The $\rho(E, N)$ generating function $Z(x, z) = \sum_{E,N}^\infty \rho(E, N)x^E z^N$ factorizes when $p = 0$ or $p = 1$

$$p = 0, \quad Z(x, z) = \prod_{i=0}^\infty \frac{1}{1 - x^{l+i}z}; \quad p = 1, \quad Z(x, z) = \prod_{i=0}^\infty (1 + x^{l+i}z) \tag{87}$$

In terms of bosons or fermions, (87) is the grand partition function for a bosonic or fermionic gas with fugacity z and, denoting $x = e^{-\beta}$, temperature $T = 1/\beta$ where

$$E = \sum_{i=0}^\infty n_i(l + i) \quad N = \sum_{i=0}^\infty n_i \tag{88}$$

with $n_i = 0, 1, 2, ...$ in the Bose case ($p = 0$) and $n_i = 0, 1$ in the Fermi case ($p = 1$). Equivalently

$$E = \sum_{i=1}^N l_i \tag{89}$$

with $l \le l_1 \le l_2 \le ... \le l_N$ (Bose) or $l \le l_1 < l_2 < ... < l_N$ (Fermi).

When p is an integer ≥ 2, (86) can be regarded as the N-body partition function of an interacting bosonic gas with the N-body spectrum

$$E = \sum_{i=1}^N l_i + pN(N - 1)/2; \quad l \le l_1 \le l_2 \le ... \le l_N \tag{90}$$

Clearly, (90) goes beyond the Fermi point $p = 1$ and describes some kind of "super-fermions". In contrast to the Bose and Fermi cases, a factorization such as (87) is not possible, due to the interacting nature of (90). One has instead the functional relation

$$Z(x, z) = Z(x, xz) + x^l zZ(x, x^p z) \tag{91}$$

which embodies the combinatorial identity

$$\rho(E, N) = \rho_0\left(E - p\frac{N(N - 1)}{2}, N\right) \tag{92}$$

where $\rho_0(E, N)$ stands for the usual partition counting.

One could push [51] this analysis further to p real positive. When $p \in [0, 1]$ and $l = 1$, one would obtain a partition problem interpolating between the usual (bosonic) one and the restricted (fermionic) one. It is manifest that, if p is replaced by $-\alpha$, the spectrum (90) coincides, under a rescaling and up to an irrelevant zero-point energy, with the N-body quantum spectrum (76) of the harmonic Calogero model. In a partition problem, one is interested in the large E and N asymptotic behavior of $\rho(E, N)$, which corresponds to the regime $x \to 1$, i.e. $\beta \to 0$. Consider the cluster expansion $\ln Z(x, z) = \sum_{n=1}^\infty b_n z^n$. In the limit $\beta \to 0$ one obtains

$$b_n = \frac{1}{\beta}\frac{e^{-nl\beta}}{n^2}\prod_{k=1}^{n-1}(1 - \frac{pn}{k}); \quad b_1 = \frac{1}{\beta}e^{-l\beta} \tag{93}$$

The limit $\beta \to 0$ should not be confused with the thermodynamic limit in quantum systems. There is no thermodynamic limit prescription like (46). Still, using (49) (with ω_c replaced by l) and taking care of the unwanted $1/n$ factor in (93), one obtains, provided that $ze^{-\beta l} < 1$,

$$\ln Z(\beta, z) = \int_l^\infty \ln y(ze^{-\beta\epsilon}) d\epsilon \tag{94}$$

with

$$y - ze^{-\beta\epsilon}y^{1-p} = 1 \tag{95}$$

This is again of the form (67) and (68), the statistical parameter $-\alpha$ being replaced by the minimal difference partition parameter p, and the 1-body density of states being the Heaviside function $\rho(\epsilon) = \theta(\epsilon - l)$. The minimal difference partition combinatorics is equivalently described, in the small β limit, by a gas of particles obeying exclusion statistics with a uniform density of states[9].

This correspondence happens to be useful technically: (94) and (95) are the building blocks of the minimal difference partition asymptotics. The average integer $\bar{E} = -\partial \ln Z(\beta, z)/\partial \beta = \int_l^\infty n\epsilon d\epsilon$ and the average number of integer parts $\bar{N} = z\partial \ln Z(\beta, z)/\partial z = \int_l^\infty n d\epsilon$, are both given in terms of $n = z\partial \ln y/\partial z$, the mean occupation number at "part" ϵ and fugacity z, which satisfies

$$ze^{-\beta\epsilon} = \frac{n}{(1 + (1-p)n)^{1-p}(1-pn)^p} \quad \text{with} \quad n \le \frac{1}{p} \tag{96}$$

One obtains

$$\bar{E} - l\bar{N} = \frac{1}{\beta} \ln Z(\beta, z) \qquad \bar{N} = \frac{1}{\beta} \ln y(ze^{-\beta l}) \tag{97}$$

so that the entropy[10] $S \equiv \ln Z(\beta, z) + \beta\bar{E} - (\ln z)\bar{N}$ rewrites as

$$S = 2\beta\left(\bar{E} - l\bar{N} - \frac{p}{2}\bar{N}^2\right) - \bar{N}\ln(1 - e^{-\beta\bar{N}}) \tag{98}$$

with

$$\bar{E} - l\bar{N} - \frac{p}{2}\bar{N}^2 = -\frac{1}{\beta^2} \int_0^{1-e^{-\beta\bar{N}}} \frac{\ln(1-u)}{u} du \tag{99}$$

Inverting (99) gives β as a function of $\bar{E}$ and $\bar{N}$ so that the entropy S in (98) becomes a function of $\bar{E}$ and $\bar{N}$ only. Doing so, one has a definite information [51] on the asymptotic behavior of $\rho(E, N) \simeq e^{S(E,N)}$ when E and N are large, and also, of $\rho(E) = \sum_{N=1}^\infty \rho(E, N)$ when E is large. One obtains a generalization of the Hardy-Ramajunan asymptotics [52] to the minimal difference partition problem. One can also obtain [53] the average limit shape of the Young diagrams associated with the minimal difference partition problem, generalizing the usual partition limit shape [54]. The limit shape at a part of height h depends solely on the statistical function y evaluated at $\epsilon = h$ and at $z = 1$

$$\beta\bar{W}_h = \ln y(e^{-\beta h}) \tag{100}$$

where β scales as $\beta^2 E = \int_0^\infty \ln y(e^{-\epsilon}) d\epsilon$.

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[9]There is no microscopic quantum Hamiltonian leading to (94) and (95).

[10]The simple expression in (97) for $\bar{N}$ is possible because of the constant density of states.

So far p being a positive integer has insured that the N-body spectrum in (90) is well defined. However, y in (95) is still meaningful when p is a negative integer. It is the $(1-p)$-ary tree generating function, so that the coefficient at order n of its expansion in powers of $ze^{-\beta\epsilon}$ as given in (69) (with $-\alpha$ replaced by p) is the number of ways to build a $(1-p)$-ary tree with n nodes. For example, at $p = -1$, y generates the Catalan numbers associated with binary trees.

Consider, as a toy model [55], the factorized $(1-p)$-ary tree generating function

$$Z(x, z) = \prod_{i=0}^{\infty} y(zx^{l+i}) \tag{101}$$

where y satisfies (101) with $\epsilon = l + i$. (101) narrows down to (87) when $p = 0$ (Bose case). Its combinatorial interpretation is that $\rho(E, N)$ deduced from (101) counts the number of usual partitions of an integer E into N integer parts bigger or equal to l, with an additional degeneracy stemming from the $(1 - p)$-tree arborescence when, in a given partition, a part occurs n times. This enlarged degeneracy goes beyond the Bose point to define some kind of "superbosons".

One can analytically continue p to the whole negative real axis. In the large E and N limit, i.e. β smaller and smaller, one encounters a maximal temperature beyond which it is not possible to heat the system. Indeed, from (95) it follows that $y(zx^{l+i})$ in (101) obeys to $y - ze^{-\beta(l+i)}y^{1-p} = 1$, which is well defined only if [39]

$$ze^{-\beta l} < (1 - p)^{p-1}(-p)^{-p} < 1 \tag{102}$$

When $z = 1$, it defines a dimensionless "Hagedorn temperature"

$$T = \frac{l}{(1 - p)\ln(1 - p) + p\ln(-p)} \tag{103}$$

just below which E and N become large so that the asymptotic of $\rho(E, N)$ can be addressed.

5   Conclusion

In two dimensions intermediate anyonic statistics interpolating from Bose to Fermi statistics are allowed. Their definition does not involve anything else than the usual concept at the basis of quantum statistics, namely free particles endowed with particular boundary exchange conditions on their N-body wavefunctions. It happens that these boundary conditions have a much richer structure in two dimensions than in three and higher dimensions. This in turn can be understood in terms of the topology of paths in the N-particle configuration space, where non trivial braiding occurs in two dimensions, and not in higher dimensions. A flux-charge composite picture emerges to encode the braiding statistics in physical terms, via topological Aharonov-Bohm interactions and singular magnetic fields.

The anyon model as such is certainly fascinating as far as quantum mechanics is concerned, but it remains an abstract construction whose complexity is daunting. However, when projected onto the LLL of an external magnetic field, the model becomes tractable and, even more, solvable. The LLL set up is clearly adapted to the QHE and to the FQHE physics. Haldane/exclusion statistics, which can be

obtained as a LLL-anyon mean-field picture in the screening regime, leads to LLL-anyon thermodynamics.

It would certainly be rewarding if LLL anyons could be relevant experimentally, for example by uncovering some experimental hints at FQHE filling factors of the existence of quasiparticles with anyonic/exclusion statistics. Fractional charges have already been seen in shot noise FQHE experiments [56], but the nontrivial statistical nature of the charge carriers in FQHE edge currents has so far remained elusive in experiments which rely mainly on Aharonov-Bohm interferometry [28]. Note also a recent proposal for the possible experimental tracking of abelian and nonabelian anyonic statistics in Mach-Zehnder interferometers [57].

Finally, on the theoretical side, physical interactions, together with topological anyonic interactions, should also be taken into account in order to produce more realistic models.

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