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The Mathematics of String Theory

Robbert H. Dijkgraaf
Institute for Theoretical Physics &
Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Valckenierstraat 65
1018 XE Amsterdam, The Netherlands

Abstract. String theory has had a profound impact in the development of mathematics. I indicate how string theory can be considered as a two-parameter deformation of classical geometry, where one parameter controls the generalization from points to loops, and the other parameter controls the quantization in terms of the sum over topologies of Riemann surfaces. The final mathematical formulation of non-perturbative string theory, which is not yet there, will have to bring together geometry, non-commutative algebra and loop spaces.

1 Introduction

Over the years string theory [1] has been able to enrich various fields of mathematics. Subjects like algebraic and differential geometry, topology, representation theory, infinite dimensional analysis and many others have been stimulated by new concepts such as mirror symmetry [2, 3], quantum cohomology [4] and conformal field theory [5]. In fact, one can argue that this stimulating influence in mathematics will be a lasting and rewarding impact of string theory in science, whatever its final role in fundamental physics. String theory seem to be the most complex and richest mathematical object that has so far appeared in physics and the inspiring dialogue between mathematics and physics that it has triggered is blooming and spreading in wider and wider circles of mathematics.

1.1 Physics and mathematics

This synergy between physics and mathematics is definitely not a new phenomenon. Mathematics has a long history of drawing inspiration from the physical sciences, going back to astrology, architecture and land measurements in Babylonian and Egyptian times. Certainly this reached a high point in the 16th and 17th centuries with the development of what we now call classical mechanics. One of its leading architects, Galileo, has given us the famous image of the "Book of Nature" in *Il Saggiatore*, waiting to be decoded by scientists

> *Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the characters in which it is written. It is written in the language of mathematics, and its characters are triangles, circles, and other geometric figures without which it is humanly impossible to understand a single word of it; without these one is wandering in a dark labyrinth.*

This deep respect for mathematics didn't disappear after the 17th century. Again in the beginning of the last century we saw again a wonderful intellectual union of physics and mathematics when the great theories of general relativity and quantum mechanics were developed. In all the centers of the mathematical world this was closely watched and mathematicians actively participated. If anywhere this was so in Göttingen, where Hilbert, Minkowski, Weyl, Von Neumann and many other mathematicians made important contributions to physics.

Theoretical physics have always been fascinated by the beauty of their equations. Here we can even quote Feynman, who was certainly not known as a fine connaisseur of higher abstract mathematics:

To those who do not know mathematics it is difficult to get across a real feeling as to the beauty, the deepest beauty, of nature ... If you want to learn about nature, to appreciate nature, it is necessary to understand the language that she speaks in.

But then Feynman also said "If all mathematics disappeared today, physics would be set back exactly one week." (One mathematician's answer to this remark was: "This was the week God created the world.")

But despite the warm feelings of Feynman, the paths of fundamental physics and mathematics started to diverge dramatically in the 1950s and 1960s. In the struggle with all the new subatomic particles physicists were close to giving up the hope of a beautiful underlying mathematical structure. On the other hand mathematicians were very much in an introspective mode these years. Because the fields were standing back to back, Dyson famously stated in his Gibbs Lecture in 1972:

I am acutely aware of the fact that the marriage between mathematics and physics, which was so enormously fruitful in past centuries, has recently ended in divorce.

But this was a premature remark, since just at time the Standard Model was being born. This brought geometry in the form of non-Abelian gauge fields, spinors and topology back to forefront. Indeed, it is remarkable fact, that all the ingredients of the standard model have a completely natural mathematical interpretation in terms of connections, vector bundles and Clifford algebras. Soon mathematicians and physicists started to build this dictionary and through the work of Atiyah, Singer, 't Hooft, Polyakov and many others a new period of fruitful interactions between mathematics and physics was born.

1.2 Strings and mathematics

It is fair to say, I believe, that this renewed bond between mathematics and physics has been greatly further stimulated with the advent of string theory as the dominant driving force in fundamental particle physics. There is quite a history of developing and applying of new mathematical concepts in the "old days" of string theory, leading among others to representations of Kac-Moody and Virasoro algebras, vertex operators and supersymmetry. But since the seminal work of Green and Schwarz in 1984 on anomaly cancellations — twenty years ago this August – these interactions truly exploded. In particular with the discovery of Calabi-Yau manifolds as compactifications of the heterotic strings with promising phenomenological prospectives by the pioneering work of Witten many techniques of algebraic geometry entered the field.

Most of these developments have been based on the perturbative formulation of string theory, either in the Lagrangian formalism in terms of maps of Riemann surfaces into manifolds or in terms of the quantization of loop spaces. This perturbative approach is however only an approximate description that appears for small values of the quantization parameter.

Recently there has been much progress in understanding a more fundamental description of string theory that is sometimes described as M-theory. It seems to unify three great ideas of twentieth century theoretical physics and their related mathematical fields:

- General relativity; the idea that gravity can be described by the Riemannian geometry of space-time. The corresponding mathematical fields are topology, differential and algebraic geometry, global analysis.

- Gauge theory; the description of forces between elementary particles using connections on vector bundles. In mathematics this involves the notions of K-theory and index theorems and more generally non-commutative algebra.

- Strings, or more generally extended objects (branes) as a natural generalization of point particles. Mathematically this means that we study spaces primarily through their (quantized) loop spaces. This relates naturally to infinite-dimensional analysis and representation theory.

At present it seems that these three independent ideas are closely related, and perhaps essentially equivalent. To some extent physics is trying to build a dictionary between geometry, gauge theory and strings. From a mathematical perspective it is extremely interesting that such diverse fields are intimately related. It makes one wonder what the overarching structure will be.

It must be said that in all developments there have been two further ingredients that are absolutely crucial. The first is quantum mechanics – the description of physical reality in terms of operator algebras acting on Hilbert spaces. In most attempts to understand string theory quantum mechanics has been the foundation, and there is little indication that this is going to change.

The second ingredient is supersymmetry – the unification of matter and forces. In mathematical terms supersymmetry is closely related to De Rham complexes and algebraic topology. In some way much of the miraculous interconnections in string theory only work if supersymmetry is present. Since we are essentially working with a complex, it should not come to a surprise to mathematicians that there are various 'topological' indices that are stable under perturbation and can be computed exactly in an appropriate limit. Indeed it is the existence of these topological quantities, that are not sensitive to the full theory, that make it possible to make precise mathematical predictions, even though the final theory is far from complete.

1.3 Quantum invariants

Mathematics studies abstract patterns and structures. As such it has a hierarchical view of the world, where things are first put in broadly defined categories and then are more and more refined and distinguished. For instance, in topology one studies spaces in a very crude fashion, whereas in geometry the actual shape of a space matters. Two-dimensional (closed, oriented) surfaces are topologically completely determined by their genus or number of handles $g = 0, 1, 2, \ldots$. So we have a topological invariant g that associates to each surface a number

$$g : \{Surfaces\} \rightarrow \mathbb{Z}_{\geq 0}.$$

In general such invariants are very hard to come by, and quantum physics, in particular particle and string theory, has proved to be a fruitful source of inspiration for new invariants. This should perhaps not come as a complete surprise. Roughly one can say that quantum theory takes a geometric object (manifold, knot, map) and associates to it a number, often a complex number, that represents the probability amplitude that the rules of quantum mechanics associate to a certain physical process that is represented by the geometric object. For example, a knot in $\mathbb{R}^3$ can stand for the world-line of a particular particle and a manifold for a particular space-time. Once we have associated concrete numbers to geometric objects one can operate on them with various algebraic operations. In knot theory one has the concept of relating knots through recursion relations (skein relations) or even differentiation (Vassiliev invariants). In this very general way quantization can be thought of as a map

$$Geometry \rightarrow Algebra.$$

that brings geometry into the real of algebra. This often gives powerful new perspectives, as we will see in a few examples later.

1.4 String theory as a deformation of classical geometry

For pedagogical purposes in this lecture we will consider the rôle of string theory in mathematics as a two parameter family of deformations of "classical" Riemannian geometry. Let us introduce these two parameters heuristically. (We will give a more precise explanation later.)

First, in perturbative string theory we study the loops in a space-time manifold. These loops can be thought to have an intrinsic length ℓ_s, the *string length*. Because of the finite extent of a string, the geometry is necessarily "fuzzy." At least at an intuitive level it is clear that in the limit $\ell_s \rightarrow 0$ the string degenerates to a point, a constant loop, and the classical geometry is recovered. The parameter ℓ_s controls the "stringyness" of the model. We will see how the quantity $\ell_s^2 = \alpha'$ plays the role of Planck's constant on the worldsheet of the string. That is, it controls the quantum correction of the two-dimensional field theory on the world-sheet of the string.

A second deformation of classical geometry has to do with the fact that strings can split and join, sweeping out a surface Σ of general topology in space-time. According to the general rules of quantum mechanics we have to include a sum over all topologies. Such a sum over topologies can be regulated if we can introduce a formal parameter g_S, the *string coupling*, such that a surface of genus h gets weighted by a factor g_s^{2h-2}. Higher genus topologies can be interpreted as virtual processes wherein strings split and join — a typical quantum phenomenon. Therefore the parameter g_s controls the quantum corrections. In fact we can equate g_s^2 with Planck's constant in space-time. Only for small values of g_s can string theory be described in terms of loop spaces and sums over surfaces.

In fact, in the case of particles we know that for large values of g_s it is better to think in terms of waves, or more precisely quantum fields. So one could expect that for large g_s and α' the right framework is string field theory [6]. This is partly true, but it is in general difficult to analyze string field theory directly. In particular the occurrence of branes, higher-dimensional extended objects that will play an important role in the subsequent, is often obscure.

Summarizing we can distinguish two kinds of deformations: *stringy* effects parametrized by α', and *quantum* effects parametrized by g_s. This situation can be described with the following diagram

α' large	conformal field theory *strings*	M-theory *string fields, branes*
$\alpha' \approx 0$	quantum mechanics *particles*	quantum field theory *fields*
	$g_s \approx 0$	g_s large

It is perhaps worthwhile to put some related mathematical fields and leading mathematicians in a similar table

α' large	quantum cohomology (*Gromov, Witten*)	non-commutative geometry (*Connes*)
$\alpha' \approx 0$	combinatorical knot invariants (*Vassiliev, Kontsevich*)	4-manifold, 3-manifolds, knots (*Donaldson, Witten, Jones*)
	$g_s \approx 0$	g_s large

2 Quantum mechanics and particles

As a warm-up let us start by briefly reviewing the quantum mechanics of point particles in more abstract mathematical terms.

In classical mechanics we describe point particles on a Riemannian manifold X that we think of as a (Euclidean) space-time. Pedantically speaking we look at X through maps

$$x : \, pt \to X$$

of an abstract point into X. Quantum mechanics associates to the classical configuration space X the Hilbert space $\mathcal{H} = L^2(X)$ of square-integrable wavefunctions. We want to think of this Hilbert space as associated to a point

$$\mathcal{H} = \mathcal{H}_{pt}.$$

For a supersymmetric point particle, we have bosonic coordinates x^μ and fermionic variables θ^μ satisfying

$$\theta^\mu \theta^\nu = -\theta^\nu \theta^\mu.$$

We can think of these fermionic variables geometrically as one-forms $\theta^\mu = dx^\mu$. So, the supersymmetric wavefunction $\Psi(x,\theta)$ can be interpreted as a linear superposition of differential forms on X

$$\Psi(x,\theta) = \sum_n \Psi_{\mu_1\ldots\mu_n} dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_n}.$$

So, in this case the Hilbert space is given by the space of (square-integrable) de Rham differential forms $\mathcal{H} = \Omega^*(X)$.

Classically a particle can go in a time t from point x to point y along some preferred path, typically a geodesic. Quantum mechanically we instead have a linear evolution operator

$$\Phi_t : \mathcal{H} \to \mathcal{H}.$$

that describes the time evolution. Through the Feynman path-integral this operator is associated to maps of the line interval of length t into X. More precisely, the kernel $\Phi_t(x,y)$ of the operator Φ_t, that gives the probability amplitude of a particle situated at x to arrive at position y in time t, is given by the path-integral

$$\Phi_t(x,y) = \int_{x(\tau)} \mathcal{D}x \ \exp\left[-\int_0^t d\tau |\dot{x}|^2\right]$$

over all paths $x(\tau)$ with $x(0) = x$ and $x(t) = y$. Φ_t is a famous mathematical object — the integral kernel of the heat equation

$$\frac{d}{dt}\Phi_t = \Delta\Phi_t, \qquad \Phi_0 = \delta(x-y).$$

These path-integrals have a natural gluing property: if we first evolve over a time t_1 and then over a time t_2 this should be equivalent to evolving over time $t_1 + t_2$. That is, we have the composition property of the corresponding linear maps

$$\Phi_{t_1} \circ \Phi_{t_2} = \Phi_{t_1+t_2}. \tag{1}$$

This allows us to write

$$\Phi_t = e^{-tH}$$

with H the Hamiltonian. In the case of a particle on X the Hamiltonian is of course simply given by (minus) the Laplacian $H = -\Delta$. The composition property (1) is a general property of quantum field theories. It leads us to Segal's functorial view of quantum field theory, as a functor between the categories of manifolds (with bordisms) to vector spaces (with linear maps) [8].

In the supersymmetric case the Hamiltonian can be written as

$$H = -\Delta = -(dd^* + d^*d)$$

Here the differentials d, d^* play the role of the supercharges

$$d = \psi^\mu \frac{\partial}{\partial x^\mu}, \qquad d^* = g^{\mu\nu} \frac{\partial^2}{\partial\psi^\mu \partial x^\nu}.$$

The ground states of the supersymmetric quantum mechanics satisfy $H\Psi = 0$ and are therefore harmonic forms

$$d\Psi = 0, \qquad d^*\Psi = 0.$$

Therefore they are in 1-to-1 correspondence with the de Rham cohomology group of the space-time manifold

$$\Psi \in \mathrm{Harm}^*(X) \cong H^*(X).$$

We want to make two additional remarks. First we can consider also a closed 1-manifold, namely a circle S^1 of length t. Since a circle is obtained by identifying two ends of an interval we can write

$$Z_{S^1} = \mathrm{Tr}_{\mathcal{H}}\Phi_t$$

Here the partition function Z_{S^1} is a number associated to the circle S^1 that encodes the spectrum of the operator Δ. We can also compute the supersymmetric partition function by using the fermion number F (defined as the degree of the corresponding differential form). It computes the Euler number

$$\mathrm{Tr}_{\mathcal{H}}(-1)^F \Phi_t = \dim H^{even}(X) - \dim H^{odd}(X) = \chi(X)$$

Secondly, in this set-up all world-lines of the particles come with a metric, *i.e.*, a total lenth t. To make the step from the quantum mechanics to the propagation of a single particle in quantum field theory we have to integrate over this metric. In case of an interval we obtain in this way the usual propagator of a massless particle, the Greens' function of the Laplacian,

$$\int_0^\infty dt\, e^{t\Delta} = \frac{1}{\Delta} = \frac{1}{p^2}.$$

3 Conformal field theory and strings

We will now introduce our first deformation parameter α' and generalize from point particles and quantum mechanics to strings and conformal field theory.

3.1 Sigma models

A string can be considered as a parametrized loop. So, in this case we study the manifold X through maps

$$x:\ S^1 \to X$$

that is, through the free loop space $\mathcal{L}X$.

Quantization will associate a Hilbert space to this loop space. Roughly one can think of this Hilbert space as $L^2(\mathcal{L}X)$, but it is better to think of it as a quantization of an infinitesimal thickening of the locus of constant loops $X \subset \mathcal{L}X$. These constant loops are the fixed points under the obvious S^1 action on the loop space. The normal bundle to X in $\mathcal{L}X$ decomposes into eigenspaces under this S^1 action, and this gives a description (valid for large volume of X) of the Hilbert space $\mathcal{H}_{S^1}$ associated to the circle as the normalizable sections of an infinite Fock space bundle over X.

$$\mathcal{H}_{S^1} = L^2(X, \mathcal{F}_+ \otimes \mathcal{F}_-)$$

where the Fock bundle is defined as

$$\mathcal{F} = \bigotimes_{n\geq 1} S_{q^n}(TX) = \mathbb{C} \oplus qTX \oplus \cdots$$

Here we use the formal variable q to indicate the $\mathbb{Z}$-grading of $\mathcal{F}$ and we use the standard notation

$$S_q V = \bigoplus_{N\geq 0} q^N\, S^N V$$

for the generating function of symmetric products of a vector space V.

When a string moves in time it sweeps out a surface Σ. For a free string Σ has the topology of $S^1 \times I$, but we can also consider at no extra cost interacting strings that join and split. In that case Σ will be a oriented surface of arbitrary topology. So in the Lagrangian formalism one is let to consider maps

$$x:\ \Sigma \to X.$$

As is explained in the other lectures, there is a natural action for such a sigma model if we pick a Hodge star or conformal structure on Σ (together with of course a Riemannian metric g on X)

$$S(x) = \int_\Sigma g_{\mu\nu} dx^\mu \wedge *dx^\nu$$

The critical points of $S(x)$ are the harmonic maps.

In the Lagrangian quantization formalism one considers the formal path-integral over all maps $x : \Sigma \to X$

$$\Phi_\Sigma = \int_{x:\ \Sigma \to X} e^{-S/\alpha'}.$$

Here the constant α' plays the role of Planck's constant on the string worldsheet Σ. It can be absorbed in the volume of the target X by rescaling the metric as $g \to \alpha' \cdot g$. The semi-classical limit $\alpha' \to 0$ is therefore equivalent to the limit $vol(X) \to \infty$.

3.2 Functorial description

In the functorial description of conformal field theory the maps Φ_Σ are abstracted away from the sigma model definition.

Starting point is now an arbitrary (closed, oriented) Riemann surface Σ with boundary. This boundary consists of a collections of oriented circles. One declares these circles in-coming or out-going depending on whether there orientation matches that of the Σ. To a surface Σ with m in-coming and n out-going boundaries one associates a linear map

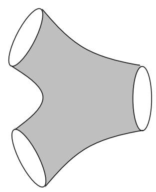
$$\Phi_\Sigma :\ \mathcal{H}_{S^1}^{\otimes n} \to \mathcal{H}_{S^1}^{\otimes n}$$

These maps are not independent but satisfy gluing axioms that generalize the simple composition law (1)

$$\Phi_{\Sigma_1} \circ \Phi_{\Sigma_2} = \Phi_\Sigma$$

where Σ is obtained by gluing Σ_1 and Σ_2 on their out-going and incoming boundaries respectively.

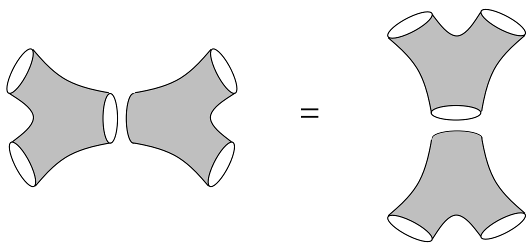
In this way we obtain what is known as a modular functor. It has a rich algebraic structure. For instance, the sphere with three holes



gives rise to a product

$$\Phi :\ \mathcal{H}_{S^1} \otimes \mathcal{H}_{S^1} \to \mathcal{H}_{S^1}$$

Using the fact that a sphere with four holes can be glued together from two copies of the three-holed sphere one shows that this product is essentially commutative and associative



Once translated in terms of transition amplitudes, these relation lead to non-trivial differential equations and integrable hierarchies. For more details see *e.g.* [4, 7].

3.3 Stringy geometry and T-duality

Two-dimensional sigma models give a natural one-parameter deformation of classical geometry. The deformation parameter is Planck's constant α'. In the limit $\alpha' \to 0$ we localize on constant loops and recover quantum mechanics or point particle theory. For non-zero α' the non-constant loops contribute.

In fact we can picture the moduli space of CFT's roughly as follows. It will have components that can be described in terms of a target spaces X. For these models the moduli parametrize Ricci-flat metrics plus a choice of B-field. These components have a boundary 'at infinity' which describe the large volume manifolds. We can use the parameter α' as local transverse coordinate on the collar around this boundary. If we move away from this boundary stringy corrections set in. In the middle of the moduli space exotic phenomena can take place. For example, the automorphism group of the CFT can jump, which gives rise to orbifold singularities at enhanced symmetry points.

The most striking phenomena that the moduli space can have another boundary that allows again for a semi-classical interpretation in terms of a second classical geometry $\hat{X}$. These points look like quantum or small volume in terms of the original variables on X but can also be interpreted as large volume in terms of a set of dual variables on a dual or mirror manifold $\hat{X}$. In this case we speak of a T-duality. In this way two manifold X and $\hat{X}$ are related since they give rise to the same CFT.

The most simple example of such a T-duality occurs for toroidal compactification. If $X = T$ is an torus, the CFT's on T and its dual T^* are isomorphic. We will explain this now in more detail. These kind of T-dualities have led to spectacular mathematical application in mirror symmetry, as we will review after that.

3.4 Particles and strings on a torus

Let us consider a particle or a string on a space-time that is given by a n-dimensional torus, written as the quotient

$$T = \mathbb{R}^n / L$$

with L a rank n lattice.

States of a quantum mechanical point particle on T are conveniently labeled by their momentum

$$p \in L^*.$$

The wavefunctions $\Psi(x) = e^{ipx}$ form a basis of $\mathcal{H} = L^2(T)$ that diagonalizes the Hamiltonian $H = -\Delta = p^2$. So we can decompose the Hilbert space as

$$\mathcal{H} = \bigoplus_{p \in L^*} \mathcal{H}_p,$$

where the graded pieces $\mathcal{H}_p$ are all one-dimensional. There is a natural action of the symmetry group

$$G = SL(n, \mathbb{Z}) = \operatorname{Aut} L$$

on the lattice $\Gamma = L$ and the Hilbert space $\mathcal{H}$. (These transformations will in general not leave the metric invariant, but instead give by pull-back another flat metric on T.)

In the case of a string moving on the torus T states are labeled by a second quantum number: their winding number

$$w \in L$$

which is simply the class in $\pi_1 T$ of the corresponding classical configuration. The winding number simply distinguishes the various connected components of the loop space $\mathcal{L}T$, since

$$\pi_0 \mathcal{L}T = \pi_1 T \cong L.$$

We therefore see a natural occurrence of the so-called Narain lattice which is the set of momenta $p \in L^*$ and winding numbers $w \in L$

$$\Gamma^{n,n} = L \oplus L^*$$

This is an even self-dual lattice of signature (n, n) with inner product

$$p = (w, k), \qquad q^2 = 2w \cdot k.$$

It turns out that all the symmetries of the lattice $\Gamma^{n,n}$ lift to symmetries of the full conformal field theory built up by quantizing the loop space. The elements of the symmetry group of the Narain lattice

$$SO(n,n,\mathbb{Z}) = \mathrm{Aut}\,\Gamma^{n,n}$$

are examples of T-dualities. A particular example is the interchange of the torus with its dual

$$T \leftrightarrow T^*.$$

T-dualities that interchange a torus with its dual can be also applied fiberwise. If the manifold X allows for a fibration $X \to B$ whose fibers are tori, then we can produce a dual fibration where we dualize all the fibers. This gives a new manifold $\widehat{X} \to B$. Under suitable circumstances this produces an equivalent supersymmetric sigma model. The symmetry that interchanges these two manifolds

$$X \leftrightarrow \widehat{X}$$

is called mirror symmetry [2, 3].

3.5 Topological strings, quantum cohomology, and mirror symmetry

In the case of point particles it was instructive to consider the supersymmetric extension since we naturally produced differential form on the target space. These differential forms are able, through the De Rahm complex, to capture the topology of the manifold. In fact, reducing the theory to the ground states, we obtained exactly the harmonic forms that are unique representatives of the cohomology groups. In this way we made the step from functional analysis and operator theory to topology.

In a similar fashion there is a formulation of string theory that is able to capture the topology of string configurations. This is called topological string theory. This is quite a technical subject, that is impossible to do justice to within the confines of this lecture, but I will sketch the essential features. For more details see *e.g.* [3]. (A crucial property of topological string theory, that again we cannot really explain here, is that besides being a beautiful and rich mathematical structure, it is also able to compute some very specific "topological" amplitudes in the full-fledged superstring and therefore also captures physical information.)

Roughly, the idea is the following. First, jus as in the point particle case, one introduces fermion fields θ^μ. Now these are considered as spinors on the two-dimensional world-sheet and they have two components $\theta^\mu_L, \theta^\mu_R$. The local action for these fermions is

$$\int d^2z\; g_{\mu\nu}(x)\left(\theta^\mu_L \frac{D\theta^\nu_L}{\partial \overline{z}} + \theta^\mu_R \frac{D\theta^\nu_R}{\partial z}\right)$$

One furthermore assumes that the target space X is (almost) complex so that one can use holomorphic local coordinates $x^i, \overline{x}^{\overline{\imath}}$ with a similar decomposition for the fermions. When complemented with the appropriate higher order terms this gives a sigma model that has $\mathcal{N} = (2,2)$ supersymmetry.

One now changes the spins of the fermionic fields to produce the topological string. This can be done in two inequivalent ways called the A-model and the B-model. Depending on the nature of this topological twisting the path-integral of the sigma model localizes to a finite-dimensional space.

The A-model restricts to holomorphic maps

$$\frac{\partial x^i}{\partial \overline{z}} = 0$$

This reduces the full path-integral over all maps from Σ into X to a finite-dimensional integral over the moduli space $\mathcal{M}$ of *holomorphic maps.* More precisely, it is the moduli space of pairs (Σ, f) where Σ is a Riemann surface and f is a holomorphic map $f:\ \Sigma \to X$. The A-model only depends on the Kähler class

$$[\omega] \in H^2(X)$$

of the manifold X.

A-model topological strings give an important example of a typical stringy generalization of a classical geometric structure. Quantum cohomology [4] is a deformation of the De Rahm cohomology ring $H^*(X)$ of a manifold. Classically this ring captures the intersection properties of submanifolds. More precisely, if we have three cohomology classes

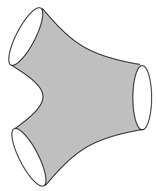
$$\alpha, \beta, \gamma \in H^*(X)$$

that are Poincaré dual to three subvarieties $A, B, C \subset X$, the quantity

$$I(\alpha, \beta, \gamma) = \int_X \alpha \wedge \beta \wedge \gamma$$

computes the intersection of the three classes A, B, and C. That is, it counts (with signs) the number of points in $A \cap B \cap C$.

In the case of the A-model we have to assume that X is a Kähler manifold or at least a symplectic manifold with symplectic form ω. Now the "stringy" intersection product is related to the three-string vertex



Mathematically it defined as

$$I_{qu}(\alpha, \beta, \gamma) = \sum_d q^d \int_{Maps_d} \alpha \wedge \beta \wedge \gamma$$

where we integrate our differential forms now over the moduli space of pseudo-holomorphic maps of degree d of a sphere into the manifold X. These maps are weighted by the classical instanton action

$$q^d = exp\left[-\frac{1}{\alpha'}\int_{S^2} \omega\right] = e^{-(d\cdot[\omega])/\alpha'}$$

Clearly in the limit $\alpha' \to 0$ only the maps of degree zero contribute, but these maps are necessarily constant and so we recover the classical definition of the intersection product by means of an integral over the space X. Geometrically, we can think of the quantum intersection product as follows: it counts the pseudo-holomorphic spheres inside X that intersect each of the three cycles A, B and C. So, in the quantum case these cycles do no longer need to actually intersect. It is enough if there is a pseudo-holomorphic sphere with points a, b, c such that $a \in A$, $b \in B$ and $c \in C$. That is, if there is a string world-sheet that connect the three.

In the B-model one can reduce to (almost) constant maps. This model only depends on the complex structure moduli of X. It most important feature is that mirror symmetry will interchange the A-model with the B-model. A famous example of the power of mirror symmetry is the original computation of Candelas et. al. [9] of the quintic Calabi-Yau manifold given by the equation

$$X: \; x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 = 0$$

in $\mathbb{P}^4$. In the case the A-model computation leads to an expression of the form

$$F(q) = \sum_d n_d \, q^d$$

where n_d computes the number of rational curves in X of degree d. These numbers are notoriously difficult to compute. The number $n_1 = 2875$ of lines is a classical result from the 19^{th} century. The next one $n_2 = 609250$ counts the different conics in the quintic and was only computed around 1980. Finally the number of twisted cubics $n_3 = 317206375$ was the result of a complicated computer program. However, now we know all these numbers and many more thanks to string theory. Here are the first ten

d	n_d
1	2875
2	6 09250
3	3172 06375
4	24 24675 30000
5	22930 59999 87625
6	248 24974 21180 22000
7	2 95091 05057 08456 59250
8	3756 32160 93747 66035 50000
9	50 38405 10416 98524 36451 06250
10	70428 81649 78454 68611 34882 49750

How are physicists able to compute these numbers? Mirror symmetry does the job. It relates the "stringy" invariants coming from the A-model on the manifold X to the classical invariants of the B-model on the mirror manifold $\widehat{X}$. In particular this leads to a so-called Fuchsian differential equation for the function $F(q)$. Solving this equation one reads off the integers n_d.

4 Non-perturbative string theory and branes

We have seen how CFT gives rise to a rich structure in terms of the modular geometry as formulated in terms of the maps Φ_Σ. To go from CFT to string theory we have to make two more steps.

4.1 Summing over string topologies

First, we want to generalize to the situation where the maps Φ_Σ are not just functions on the moduli space $\mathcal{M}_{g,n}$ of Riemann surfaces but more general differential forms. In fact, we are particular interested in the case where they are volume forms since then we can define the so-called string amplitudes as

$$A_g = \int_{\mathcal{M}_g} \Phi_\Sigma$$

This is also the general definition of Gromov-Witten invariants [4] as we will come to later. Although we suppress the dependence on the CFT moduli, we should realize that the amplitudes A_g (now associated to a *topological* surface of genus g) still have (among others) α' dependence.

Secondly, it is not enough to consider a string amplitude of a given topology. Just as in field theory one sums over all possible Feynman graphs, in string theory we have to sum over all topologies of the string world-sheet. In fact, we have to ensemble these amplitudes into a generating function.

$$A(g_s) \approx \sum_{g\geq 0} {g_s}^{2g-2} A_g .$$

Here we introduce the so-called string coupling constant g_s. Unfortunately, in general this generating function can be at best an asymptotic series expansion of an analytical function $A(g_S)$. A rough estimate of the volume of $\mathcal{M}_g$ shows that typically

$$A_g \sim 2g!$$

so the sum over string topologies will not converge. Indeed, general physics arguments tell us that the *non-perturbative* amplitudes $A(g_s)$ have corrections of the form

$$A(g_s) = \sum_{g\geq 0} {g_s}^{2g-2} A_g + \mathcal{O}(e^{-1/g_s})$$

Clearly to approach the proper definition of the string amplitudes these non-perturbative corrections have to be understood.

4.2 Non-perturbative string theory

As will be reviewed at much greater length in the other lectures, the last years have seen remarkable progress in the direction of developing such a non-perturbative formulation. Remarkable, it has brought very different kind of mathematics into the game. It involves two remarkable new ideas.

1. String theory is not a theory of strings. It is simply not enough to consider loop spaces and their quantization. We should also include other extended objects, collectively known as branes. One can try to think of these objects as associated to more general maps $Y \to X$ where Y is a higher-dimensional space. But the problem is that there is not a consistent quantization starting from 'small' branes along the lines of string theory, that is, an expansion where we control the size of Y (through α') and the topology (through g_s). However, through the formalism of D-branes [10] these can be analyzed exactly in string perturbation theory. D-branes give contribution that are of order

$$e^{-1/g_s}$$

and therefore complement the asymptotic string perturbation series.

2. As we stressed, the amplitudes A depend on many parameters or moduli. Apart from the string coupling g_s all other moduli have a geometric interpretation, in terms of the metric and B-field on X. The second new ingredient is the insight that string theory on X with string coupling g_s can be given a fully geometric realization in terms of a new theory called M-theory on the manifold $X \times S^1$, where the length of the circle S^1 is g_s [11].

4.3 Branes on a torus

If we move to the full non-perturbative string theory on a torus the story becomes more complicated then we saw in section 3.4. The lattice of quantum numbers of the various objects becomes larger and so do the symmetries. For small values of the dimension n of the torus T ($n \geq 4$) it turns out that the non-perturbative charge lattice M can be written as as the direct sum of the Narain lattice (the momenta and winding numbers of the strings) together with a lattice that keeps track of the homology classes of the branes

$$M = \Gamma^{n,n} \oplus H^{even/odd}(T)$$

Here we note that the lattice of branes (which are even or odd depending on the type of string theory that we consider)

$$H^{even/odd}(T) \cong \bigwedge\nolimits^{even/odd} L^*$$

transform as half-spinor representations under the T-duality group $SO(n,n,\mathbb{Z})$. The full duality group turns out to be the exceptional group over the integers

$$E_{n+1}(\mathbb{Z}).$$

The lattice M will form an irreducible representation for this symmetry group. These so-called U-dualities will therefore permute strings with branes.

So we see that our hierarchy

$$\{Particles\} \subset \{Strings\} \subset \{Branes\}$$

is reflected in the corresponding sequence of symmetry (sub)groups

$$SL(n,\mathbb{Z}) \subset SO(n,n,\mathbb{Z}) \subset E_{n+1}(\mathbb{Z})$$

of rank $n-1$, n and $n+1$ respectively. It is already a very deep (and generally unanswered) question what the 'right' mathematical structure is associated to a n-torus that gives rise to the exceptional group $E_{n+1}(\mathbb{Z})$.

5 D-branes

The crucial ingredient to extend string theory beyond perturbation theory are D-branes [10]. From a mathematical point of view D-branes can be considered as a relative version of Gromov-Witten theory. The starting point is now a pair of relative manifolds (X, Y) with X a d-dimensional manifold and $Y \subset X$ closed. The string worldsheets are defined to be Riemann surfaces Σ with boundary $\partial\Sigma$, and the class of maps $x : \Sigma \to X$ should satisfy

$$x(\partial\Sigma) \subset Y$$

That is, the boundary of the Riemann surfaces should be mapped to the subspace Y.

Note that in a functorial description there are now two kinds of boundaries to the surface. First there are the time-like boundaries that we just described. Here we choose a definite boundary condition, namely that the string lies on the D-brane Y. Second there are the space-like boundaries that we considered before. These are an essential ingredient in any Hamiltonian description. On these boundaries we choose initial value conditions that than propagate in time. In closed string theory these boundaries are closed and therefore a sums of circles. With D-branes there is a second kind of boundary: the open string with interval $I = [0, 1]$.

The occurrence of two kinds of space-like boundaries can be understood because there are various ways to choose a 'time' coordinate on a Riemann surface with boundary. Locally such a surface always looks like $S^1 \times \mathbb{R}$ or $I \times \mathbb{R}$. This ambiguity how to slice up the surface is a powerful new ingredient in open string theory.

To the CFT described by the pair (X, Y) we will associate an extended modular category. It has two kinds of objects or 1-manifolds: the circle S^1 (the closed string) and the interval $I = [0, 1]$ (the open string). The morphisms between two 1-manifolds are again bordisms or Riemann surfaces Σ now with a possible boundaries. We now have to kinds of Hilbert spaces: closed strings $\mathcal{H}_{S^1}$ and open strings $\mathcal{H}_I$.

Semi-classically, the open string Hilbert space is given by

$$\mathcal{H}_I = L^2(Y, \mathcal{F})$$

with Fock space bundle

$$\mathcal{F} = \bigotimes_{n \geq 1} S_{q^n}(TX)$$

Note that we have only a single copy of the Fock space $\mathcal{F}$, the boundary conditions at the end of the interval relate the left-movers and the right-movers. Also the fields are sections of the Fock space bundle over the D-brane Y, not over the full space-time manifold X. In this sense the open string states are localized on the D-brane.

5.1 Branes and matrices

One of the most remarkable facts is that D-branes can be given a multiplicity N which naturally leads to a non-Abelian structure [12].

Given a modular category as described above there is a simple way in which this can be tensored over the $N \times N$ hermitian matrices. We simply replace the Hilbert space $\mathcal{H}_I$ associated to the interval I by

$$\mathcal{H}_I \otimes Mat_{N\times N}$$

with the hermiticity condition

$$(\psi \otimes M_{IJ})^* = \psi^* \otimes M_{JI}$$

The maps Φ_Σ are generalized as follows. Consider for simplicity first a surface Σ with a single boundary C. Let C contain n 'incoming' open string Hilbert spaces with states $\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n$. These states are now matrix valued. Then the new morphism is defined as

$$\Phi_\Sigma(\psi_1 \otimes M_1, \ldots, \psi_n \otimes M_n) = \Phi_\Sigma(\psi_1, \ldots, \psi_n)\mathrm{Tr}\,(M_1 \cdots M_n).$$

In case of more than one boundary component, we simply have an additional trace for every component.

In particular we can consider the disk diagram with three open string insertions. By considering this as a map

$$\Phi_\Sigma : \mathcal{H}_I \otimes \mathcal{H}_I \to \mathcal{H}_I$$

we see that this open string interaction vertex is now given by

$$\Phi_\Sigma(\psi_1 \otimes M_1, \psi_2 \otimes M_2) = (\psi_1 * \psi_2) \otimes (M_1 M_2).$$

So we have tensored the associate string product with matrix multiplication.

If we consider the geometric limit where the CFT is thought of as the semi-classical sigma model on X, the string fields that correspond to the states in the open string Hilbert space $\mathcal{H}_I$ will become matrix valued fields on the D-brane Y, *i.e.* they can be considered as sections of $\mathrm{End}(E)$ with E a (trivial) vector bundle over Y.

This matrix structure naturally appears if we consider N different D-branes $Y_1, \ldots, Y_N$. In that case we have a matrix of open strings that stretch from brane Y_I to Y_J. In this case there is no obvious vector bundle description. But if all the D-branes coincide $Y_1 = \ldots = Y_N$ a $U(N)$ symmetry appears.

5.2 D-branes and K-theory

The relation with vector bundles has proven to be extremely powerful. The next step is to consider D-branes with *non-trivial* vector bundles. It turns out that these configurations can be considered as a composite of branes of various dimensions [13]. There is a precise formula that relates the topology of the vector bundle E to the brane charge $\mu(E)$ that can be considered as a class in $H^*(X)$. (For convenience we consider first maximal branes $Y = X$.) It reads [14]

$$\mu(E) = ch(E)\widehat{A}^{1/2} \in H^*(X). \tag{2}$$

Here $ch(E)$ is the (generalized) Chern character $ch(E) = \mathrm{Tr}\,\exp(F/2\pi i)$ and $\widehat{A}$ is the genus that appears in the Atiyah-Singer index theorem. Note that the D-bane charge can be fractional.

Branes of lower dimension can be described by starting with two branes of top dimension, with vector bundles E_1 and E_2, of opposite charge. Physically two such branes will annihilate leaving behind a lower-dimensional collection of branes. Mathematically the resulting object should be considered as a virtual bundle $E_1 \ominus E_2$ that represents a class in the K-theory group $K^0(X)$ of X [15]. In fact the map μ in (2) is a well-known correspondence

$$\mu : K^0(X) \to H^{even}(X)$$

which is an isomorphism when tensored with the reals. In this sense there is a one-to-one map between D-branes and K-theory classes [15]. This relation with K-theory has proven to be very useful.

5.3 Example: the index theorem

A good example of the power of translating between open and closed strings is the natural emergence of the index theorem. Consider the cylinder $\Sigma = S^1 \times I$ between two D-branes described by (virtual) vector bundles E_1 and E_2. This can be seen as closed string diagram with in-state $|E_1\rangle$ and out-state $|E_2\rangle$

$$\Phi_\Sigma = \langle E_2, E_1 \rangle$$

Translating the D-brane boundary state into closed string ground states (given by cohomology classes) we have

$$|E\rangle = \mu(E) \in H^*(X)$$

so that

$$\Phi_\Sigma = \int_X ch(E_1)ch(E_2^*)\widehat{A}$$

On the other hand we can see the cylinder also as a trace over the open string states, with boundary conditions labeled by E_1 and E_2. The ground states in $\mathcal{H}_I$ are sections of the Dirac spinor bundle twisted by $E_1 \otimes E_2^*$ This gives

$$\Phi_\Sigma = \mathrm{Tr}_{\mathcal{H}_I}(-1)^F = \mathrm{index}(D_{E_1 \otimes E_2^*})$$

So the index theorem follows rather elementary.

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