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The symphony of gravity

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Abstract.
The groundbreaking detection of gravitational waves on 14 September 2015 ushered in a new era in our exploration of one of nature's most profound fundamental forces: gravity. With gravitational waves now routinely detected, we have gained a novel observational framework to test gravity and investigate the properties of their sources. We gain new observational windows of the gravitational force.

In this text, we describe an innovative analytical computational framework for the inspiral phase of binary systems, achieved by embedding Einstein's theory of gravity into a quantum gravity effective action. Einstein's gravity is the leading contribution of a quantum gravity effective field theory. This framework relies on the separation of scales between long-range and short-distance physics. This setup provides a novel approach to tackling the intricate challenges of gravitational radiation. This approach provides a framework for incorporating derivations from, including quantum gravity corrections opening possibility of constraining quantum gravity corrections to classical observables.

1 Dancing massive bodies

> Les physiciens disent des trous noirs qu'à force de se concentrer dans le ciel nocturne, il leur arrive d'enrouler, dans la substance ténébreuse, l'espace qu'ils épanchent dans le temps.
>
> *Pascal Quignard*
> *La barque silencieuse Chap XXV Extase et enstase*

Our conception of the force of gravity conditions our vision of the shape and dynamic of our observable Universe. The current cosmological paradigm relies on General Relativity, Einstein's theory of gravity [1,2] – crafted as a relativistic theory of curved space-time. Gravity is inferred universal – with dynamics that couple equally to all type of matter and energy. Einstein proposed three classical tests of this theory of gravity [3]: the perihelion precession of Mercury, the deflection of light by the Sun and the gravitational redshift of light. These classical tests illustrate the ubiquity of the gravitational force. In addition, several modern tests are routinely performed to test general relativity as well as effects that, in principle, could occur in a theory of gravitation different from Einstein's theory of gravity [4]

Unfortunately, the weakness of the gravitational force compared to the other forces of Nature makes difficult to perform these tests. Detecting signals for deviations from Einstein's theory of gravity in gravitational effects is very challenging. For a long time, there was a huge observational gap in the scales where gravity could be measured with precision between the solar system range, like the test of the equivalence principle [5] and the cosmological range [6] as shown in this figure from the ESA Fundamental Physics Road-map of 2010 [7,8].

The detection of gravitational waves from binary systems has revolutionized our ability to probe gravity. Before discussing how this data has changed our understanding of gravity, let us review some highlights of the long journey from Einstein's conception of general relativity in 1915 to the detection of gravitational waves in 2015.

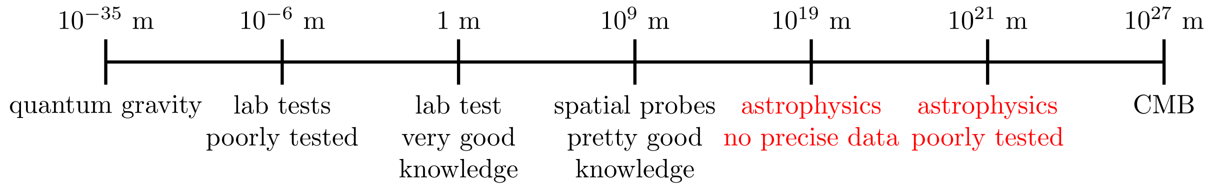

Figure 1: How well do we test gravity in 2010 adapted from fig. 2 in [7] and [8].

1.1 Gravitational radiation

Einstein's theory of gravity is a broader interpretation of Newton's classical dynamics. When dealing with small masses and low velocities, the two-body gravitational interaction described by general relativity approximates to Newton's inverse-square law. Constrained by the finite speed of light, general relativity predicts the existence of gravitational radiation.

Prior to Einstein's conception of general relativity, Henri Poincaré theorized that the gravitational force is not instantaneous but rather propagates at the speed of light. This implies a time delay, known as retardation, between any alteration in gravity and its subsequent effect. Poincaré explicitly stated that these changes are transmitted by gravitational waves, which he termed *ondes gravifiques*, though he didn't delve into their specific form.

In the article [9] he wrote in the last section "Hypothesis on Gravitation" that[1]

> Il importait d'examiner cette hypothèse de plus près et en particulier de rechercher quelles modifications elle nous obligerait à apporter aux lois de la gravitation. C'est ce que j'ai cherché à déterminer; j'ai été d'abord conduit à supposer que la propagation de la gravitation n'est pas instantanée, mais se fait avec la vitesse de la lumière. (...)
>
> Quand nous parlerons donc de la position ou de la vitesse du corps attirant, il s'agira de cette position ou de cette vitesse à l'instant où l'onde gravifique est partie de ce corps; quand nous parlerons de la position ou de la vitesse du corps attiré, il s'agira de cette position ou de cette vitesse à l'instant où ce corps attiré a été atteint par l'onde gravifique émanée de l'autre corps; il est clair que le premier instant est antérieur au second.

This is precisely the retardation, the fact that the force is not instantaneous, that field theory needs to produce gravitational waves, from the acceleration of massive body.

In 1916, Einstein realized that gravitational radiation emitted by an electron orbiting a nucleus would affect the stability of the atom in the same way that classical electromagnetic radiation does in [10].

> Gleichwohl müßten die Atome zufolge der inneratomischen Elektronenbewegung nicht nur elektromagnetische, sondern auch Gravitationsenergie

[1] An english translation is "It was important to examine this hypothesis more closely, and in particular to investigate what modifications it would require us to make to the laws of gravitation. This is what I sought to determine; I was first led to suppose that the propagation of gravitation is not instantaneous, but takes place at the speed of light. (...)

Therefore, when we speak of the position or velocity of the attracting body, we will be referring to that position or velocity at the instant when the gravitational wave left that body; when we speak of the position or velocity of the attracted body, we will be referring to that position or velocity at the instant when that attracted body was reached by the gravitational wave emitted by the other body; it is clear that the first instant is prior to the second."

> ausstrahlen, wenn auch in winzigem Betrage. Da dies in Wahrheit in der Natur nicht zutreffen dürfte, so scheint es, daß die Quantentheorie nicht nur die Maxwellsche Elektrodynamik, sondern auch die neue Gravitationstheorie wird modifizieren müssen[2]

He made the same claim in his famous 1918 paper [11] where he derived the quadrupole formula that

> Aus (27) ist ersichtlich, daß die Ausstrahlung in keiner Richtung negativ werden kann, also sicher auch nicht die totale Ausstrahlung. Bereits in der fritheren Abhandlung ist betont geworden, daß das Endergebnis dieser Betrachtung, welches einen Energieverlust der Körper infolge der thermischen Agitation verlangen würde, Zweifel an der allgemeinen Gültigkeit der Theorie hervorrufen muß. Es scheint, daß eine vervollkommnete Quantentheorie eine Modifikation auch der Gravitationstheorie wird brigen müssen.[3]

Einstein was troubled by idea that the atom could not be stable due to emission of gravitational radiation. The electromagnetic radiation of the electron orbiting around the nucleus leads to a collapse of the atom in $10^{-10}\,s$ whereas the collapse under gravitational radiation is of the order of $10^{37}\,s$. The important difference in the time is due to huge weakness of the gravitational force compared to the electromagnetic force. Something that has been a challenge for detecting the classical gravitational radiation from massive binaries and is at the heart of the difficulties of find experimental signature of quantum gravity. Therefore, Einstein motivation was not of an empirical but more of a theoretical nature based on an analogy with electrodynamics.

The quantization of gravity was carried out by Feynman [13] and DeWitt [14–16] in the 1960s. By modern standards, the quantization itself is not problematic. However, subsequent investigations showed that the high-energy behaviour of quantum corrections did not resemble that of renormalizable field theories. The question of the quantum nature of gravity is still an open subject because of a lack of experimental evidence of quantum gravity effects, mainly due to the weakness of the gravitational force.[4]

Still we assume that a quantum theory of gravity exists and that the force of gravity is mediated by the graviton a massless spin 2 particle [17]. Using idea from effective field theories [18,19], we present an approach to classical gravitational radiation by computing quantum gravity scattering amplitudes. This approach has brought a new eye on some important questions about gravitational radiation. While the quantization of gravity is primarily employed as a computational tool to explore classical radiation effects in line with the suggestion by Kovàcs and Thorne in [20], but we will argue that one can as well safely predict long range low-energy (infrared)

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[2] An english translation could read "However, according to the inner-atomic electron movement, atoms would have to emit not only electromagnetic, but also gravitational energy, even if in a tiny amount. Since this is unlikely to be the case in nature, it seems that quantum theory will have to modify not only Maxwell's electrodynamics, but also the new gravitational theory."

[3] English translation from [12] "One sees from (27) that the emission cannot turn negative in any direction; consequently, the total emission certainly cannot turn negative, either. It has already been emphasized in a previous paper that the end result of this investigation— which would require a loss of energy of bodies due to the thermal agitation—must raise doubts as to the general validity of the theory. It seems that a more complete quantum theory would also have to bring about a modification of the theory of gravitation."

[4] The question of the very short distance high-energy (ultraviolet) regime of quantum gravity is still an open question which will be discussed by Zvi Bern in this volume.

quantum gravity effects correcting the classical contribution from Einstein theory. The validity of this approach is not proof of the need to quantize gravity however, the formalism provides a framework that could potentially lead to a method for observing the quantization of gravity from long-range infrared contributions, which will be discussed in Section 6.

1.2 Rising the classical gravitational waves

In 1918, Einstein derived his famous quadrupole formula [11], which gives the rate at which gravitational waves are emitted from a system of masses based on the change in the (mass) quadrupole moment. However, Einstein wrote that it was unlikely that anyone would ever find a system whose behaviour would be measurably influenced by gravitational waves. He was pointing out that the waves from a typical binary star system would carry away so little energy that we would never even notice that the system had changed.

In General relativity gravitational waves are waves of the intensity of gravity that are generated by the accelerated masses of binary stars and other motions of gravitating masses, and propagate as waves outward from their source at the speed of light.

In 1936, Einstein and Rosen submitted an article to Physical Review titled “Do Gravitational Waves Exist?” in which they claimed to have a proof that general relativity does not allow for exact gravitational wave solutions because any such solution of the field equations would have a singularity. This paper was rejected by the renowned cosmologist Howard P. Robertson, who pointed out a mistake and suggested some revisions. Einstein and Rosen’s revised work, titled “On Gravitational Waves,” appeared in the Journal of the Franklin Institute. In this paper, they rigorously proved the existence of cylindrical gravitational waves without singularities for Euclidean space [21]. A rigorous mathematical proof of the existence of gravitational waves in Einstein theory was given by Choquet-Bruhat in 1952 [22] and [23]. Choquet-Bruhat’s local theorem of 1952 was a breakthrough and has since been fundamental for further investigations of the Cauchy problem and proved crucial to the possibility of numerically simulating the motion and gravitational radiation of coalescing binary black holes. A survey of the mathematical work on the general relativity can be found in [24].

The quadrupole formula quantifies the total power emitted from a binary system. This prediction was confirmed by observations of the energy loss due to gravitational radiation in the Hulse-Taylor binary pulsar, PSR 1913+16, discovered in 1974. The orbit of this binary system has decayed since it was discovered, in precise agreement with Einstein’s quadrupole formula to within 0.2% precision [4, 25]. This system provided an important confirmation of the existence of gravitational waves, but it only provides a weak-field test of Einstein’s theory. The lifetime to final inspiral due to the emission of gravitational radiation is calculated to be 300 million years.

On September 14, 2015, the LIGO Scientific Collaboration detected a transient gravitational wave signal produced by a binary black hole system with a total mass of approximately 65 solar masses, about 1 billion years ago, prior to impact [26]. This detection occurred 100 years after Einstein predicted the existence of gravitational waves in general relativity. The detection of the complete signal merger and ringdown has opened a new window on the physics of gravity.

The direct detection of gravitational waves from a binary black hole system is a major confirmation of two of Einstein's theory gravity most significant predictions: gravitational waves and black holes. This direct detection finally puts to rest Einstein's own doubts about the physical significance of gravitational waves and the reality of black holes.

The routine detection of gravitational waves now opens up a completely new way to analyse the properties of black holes and probe gravity in our Universe at scales that were previously inaccessible.

Subrahmanyan Chandrasekhar, a Nobel Prize winner in physics, said that "*the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time.*" Although the solutions for black holes were found shortly after the inception of general relativity, with the Schwarzschild solution given in 1916 and the Reissner and Nordström solutions given in 1916 and 1918, respectively, Einstein doubted the physical reality of black holes.

In his 1939 paper on the Schwarzschild solution [27], Einstein asked whether "*it is possible to build up a field containing such singularities with the help of the actual gravitating masses, or whether such regions with vanishing g_{44} do not exist in cases which have physical reality.*"

It was not until the 1950s, with the work of Robert Oppenheimer and John Wheeler, that black holes began to be seriously considered as possible astrophysical objects existing in our Universe. Subrahmanyan Chandrasekhar (Nobel Prize in Physics 1983) and Roger Penrose (Nobel Prize in Physics 2020) explained that sufficiently massive stars at the end of their lives, having depleted their nuclear fuel, must gravitationally collapse to become a spacetime singularity.

We have progressively obtained hints of the presence of black holes from their effects on the surrounding matter. For example, the presence of Sagittarius A*, the supermassive black hole at the centre of our galaxy, was suspected since the detection of its radio emission in February 1974, but it was difficult to observe because it is concealed by a multilayered veil. Its presence was only confirmed by the images from the *Event Horizon Telescope* collaboration in 2022 [28].

We now estimate that there are more than 100 million black holes of a solar mass in our galaxy and at least 100 billion supermassive black holes of mass of a million or a billion solar masses in the universe. We also believe that a black hole is created every second in a supernova event and that primordial (quantum) black holes were created after the Big Bang.

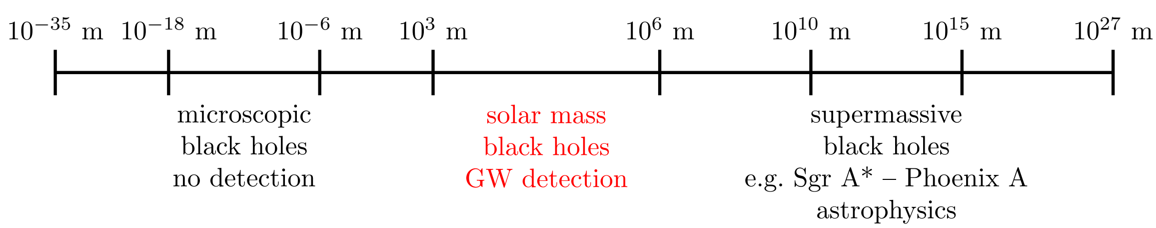


Figure 2: Sizes of Black Holes Throughout the Universe.

All of this evidence for black holes was based on their static properties and their influence on their environment. The detection of gravitational waves emitted by a binary black hole system was the first direct detection of the dynamics of black holes [26]. By analysing gravitational wave events, we can reconstruct the physical properties (mass and spin) and location in space and time of black holes, and follow the dynamics of the inspiral regime until the merging of the binary system.

1.3 Riding the gravitational waves

The gravitational waves emitted by compact binary systems, composed of neutron stars and black holes are routinely detected by the LIGO, VIRGO and KAGRA detector. And with the launch of LISA we will be able to detect them in space before their reach the Earth. The loss of gravitational energy results in a decrease of the orbital separation of the binaries and an increase of the orbital frequency. As the binaries get closer the frequency of the gravitational-waves becomes higher pitched, the signal chirps, until the two objects collide and merge.

These detections are important because we can compare the way the gravitational waves propagate from the source to the detector with the other emissions, such as X-ray and radio signals. This is a powerful tool for studying gravity at different scales, including the dynamics of black holes and their interactions with their surroundings.

The analysis of gravitational wave data from binary systems will precisely fill the gap in testing the law of gravity that was presented in the 2010 roadmap in Fig. 1.

The detection of the first gravitational wave signal has opened a new era of precision gravity. Gravitational waves can tell us about gravity at various scales, including the dynamics of black holes. This will ultimately tell us how well we understand gravity in both the weak and strong coupling regimes.

The binary mergers detected so far by LIGO-Virgo are clean sources of gravitational waves [29, 30], and the gravitational wave signal is currently modelled by general relativity in vacuum to second order in the mass ratio parameter. However, next-generation gravitational wave detectors, such as the Laser Interferometer Space Antenna (LISA), will be influenced by the environment of the sources, and the signal will be more "dirty" [31].

The lack of clear predictions for non-linearities (for example, from the accretion disk) in the post-merger phase means that these could be confused with modifications of the signal predicted by theories beyond general relativity. To accurately interpret the data from these new detectors, we need to produce accurate theoretical gravitational waveform templates and understand the limitations of our current theoretical understanding of gravity. In particular, we need to know how much we can learn about gravity in the weak and strong coupling regimes, and whether we can use gravitational wave observations to detect evidence of new physics beyond general relativity, such as modified gravity theories or quantum effects.

The profile of a gravitational wave depends on the physics of the merger and how the wave has propagated in space. There are three typical regimes for the merger:

- Inspiral phase: The two binaries are far apart with relatively slow motion and weak gravitational interaction.

- Merger phase: The binaries are close together and the gravitational interaction is strong.

- Ring-down phase: The final state of the system is relaxing.

After the binary system has merged into a black hole, the ring-down phase tells us about the structure of the black hole's horizon. We estimate that there are more than 100 million black holes in our galaxy and 100 billion supermassive black holes in the universe. These numbers indicate the potential for gravitational wave detections from black hole binary mergers at intermediate distances in the universe.

For neutron star binary mergers, we can compare the propagation of the gravitational waves to the radio signals and visible emissions. The analysis of gravitational waves will allow for precision tests of general relativity and the black hole paradigm itself. The combined gravitational wave and electromagnetic signals will allow us to determine the properties of the binary and its environment.

The recently enhanced observatories (LIGO, Virgo, and KAGRA) and the vastly improved sensitivity of the third generation of gravitational wave observatories (the Einstein Telescope, the Cosmic Explorer, and the future space-based LISA) will permit detailed measurements of the sources' physical parameters and complement, in a multi-messenger approach, the observation of signals emitted by cosmological sources obtained through other kinds of telescopes [31, 32].

1.4 Reading the gravitational waves

General relativity is not enough to explain all the observed properties of our universe. It fails to explain the observed dynamics and stability of galaxies or the accelerated expansion of the universe. Dark matter and dark energy, of unknown nature, have been introduced to quantify our lack of understanding.

"*We wouldn't call it a tension or a problem, but rather a crisis,*" commented the Nobel Prize winner David Gross, about the different measurements of the expansion rate of the Universe from local and cosmological observations [33, 34]. The signal contains information about the sources and how the gravitational waves have propagated in space and time till their detections on Earth. The knowledge of the mass distribution of sources of gravitational waves can be used to infer cosmological parameters in the absence of redshift measurements obtained from electromagnetic observations [35–37]. This is a strong invitation to consider modifications of the law of gravity over large astrophysical scales.

Black holes and compact stars and gravitational waves are amongst the most spectacular predictions of general relativity. It is therefore natural to use them as probes of the most fundamental principles of Einstein's theory [38, 39]. The gravitational-wave event constrain a plethora of mechanisms associated with the generation and propagation of gravitational waves, including the activation of scalar fields [40], gravitational leakage into large extra dimensions [41], the variability of Newton's constant [42, 43], the speed of gravitational waves, their propagation [44], gravitational Lorentz violation and the strong equivalence principle [45] and higher-derivative corrections [46]. The LIGO Scientific Collaboration and the Virgo Collaboration have verified that the observations are consistent with Einstein's theory of gravity, constraining the presence of certain parametric anomalies in the signal [37]. However, the true potential for gravitational-wave detections to both rule out exotic objects and constrain physics beyond General Relativity is severely limited by the lack of understanding of the coalescence regime in almost all relevant modified gravity theories.

Current predictions for gravitational-wave signals are based on a variety of complementary theoretical approaches: the weak field and small velocity expansion in the inspiral regime, numerical relativity for the merger, and the quasi-normal modes for the relaxation of the final black hole. They have been used for analysing successful detections, but they have their limitations. Astrophysical evidence suggests that black holes can have a variety of intrinsic angular momenta, including close to maximally allowed values. The presence of spin can lead to qualitative changes in the dynamics of a binary system, such as the orbital-plane precession when the spins are not aligned with the orbital angular momentum. Such an effect would lead, in particular, to a modulation of the amplitude, frequency and phase of the observed gravitational-wave signal.

For this, we need to produce accurate theoretical gravitational waveform templates. We have to clarify how much can be understood from exact theoretical computations. We have to answer the questions about how much we understand gravity in the weak and strong coupling regimes. And whether we can learn about gravitational physics beyond Einstein gravity.

1.5 Dynamics of massive binaries

Einstein's theory of gravity minimal coupled to matter fields is given by the Einstein-Hilbert action

$$\mathcal{S}_{\rm EH} = \int d^4x\sqrt{-g}\left[\frac{c^3}{16\pi G_N}R + g^{\mu\nu}T^{\rm matter}_{\mu\nu}\right]. \tag{1}$$

which encodes the classical dynamics of the space-time metric $g_{\mu\nu}$ with $\mu,\nu = 0,1,2,3$. There $g = \det(g)$ and R is the Ricci curvature (see [47, 48] for instance)

$$R = g^{\mu\nu}g^{\lambda\kappa}R_{\mu\nu\lambda\kappa}, \qquad R^\lambda{}_{\mu\nu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial x^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial x^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\nu_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta} \tag{2}$$

with the Christoffel symbols $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\left(\frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right)$, and $T^{\rm matter}_{\mu\nu}$ is the stress-tensor for matter minimally coupled to the metric. We will be using the mostly minus metric $(+---)$. The strength of the gravitational force is set by Newton constant G_N. The speed of light is c which we will set to 1 from now.

The variation of this action with respect to the metric gives (see [47, 48] for instance)

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G_N T^{\rm matter}_{\mu\nu} \tag{3}$$

where the Ricci tensor is given by $R_{\mu\nu} = R_{\mu\nu\lambda\kappa}g^{\lambda\kappa}$.

Solving this equation of motion is a difficult task because of the non-linearity of the gravitational interaction. The Schwarzschild black hole solution is an exact solution in four dimensions in vacuum with $T^{\rm matter}_{\mu\nu} = 0$ and reads in the standard form

$$ds^2 = \left(1 - \frac{2G_NM}{r}\right)dt^2 - \left(1 - \frac{2G_NM}{r}\right)^{-1}dr^2 - r^2(d\theta + (\sin(\theta))^2 d\phi^2). \tag{4}$$

We will show how to recover this metric using effective field theory methods in Section 4.

Two massive binaries (black holes or neutron stars) will attract gravitationally until they will merge. During the inspiral phase the system slowly loses energy through gravitational radiation. In this regime the kinetic energy is of the same order as potential energy so that $v^2 \sim G_N M/r$ where v is the relative velocity between the two binaries, r the relative distance and M the total mass of the system. One can solve the equation of motion in perturbation in powers G_N and $v/c \ll 1$ and computes the deviation from Newtonian dynamics by treating the sources non-relativistically. This is the post-Newtonian expansion initiated in [49] (see [48, §9] for some introduction). Although this historical approach has been pushed to quite some high order in G_N and the relative velocity v/c [50–53]. This approach has been described by Damour's contribution to a previous Séminaire Poincaré [54].

This approach has some important shortcomings. The fact that one performs a small relative velocity expansion $v \ll c$ forbids studying the ultra relativistic regime where $v \simeq c$ which brings crucial insight on the nature of the gravitational radiation (see for instance for a recent review on the modern approach [55] to this question). Another crucial issue is the treatment of the diverging integrals. Some divergences arise from the point-like approximation of the source and are familiar ultraviolet problems, but these divergences are mixed with infrared divergences from the propagation long range effects of the gravitational interaction.

For the third generation of gravitational waves detectors the majority of the cycles in the detector's band will occur during the inspiral phase. Therefore, building a framework that delivers analytic expressions for the classical gravitational observables in a post-Minkowskian expansion to high-loop orders is very much needed.

Different lines of approaches based on effective field theories have been proposed for computing analytically the inspiral phase the binary systems. A first approach is the non-relativistic general relativity formalism of [56–58] that uses an effective field theory to model the gravitationally bound binary system as point-like massive particles coupled to the gravitational field. This approach valid for widely separated massive objects can have spinning objects. Integrating out the suitably non-relativistically decomposed graviton field in the path integral yields a Feynman diagrammatic expansion for the classical effective potential of the binary system and associated gravitational radiation [52, 56–59]. This is a reformulation of the post-Newtonian expansion inspired by techniques and method effective field theory in particle physics (in particular the small velocity expansion for computing the potential between in heavy quarks in QCD). This approach has led to a clarification in the treatment of the short distance (ultraviolet) and long distance (infrared) divergences in the post-Newtonian expansion thanks to the natural separation of scales in the problem.

These methods have seen a rapid development thanks to the adaptation of methods for evaluating amplitude in particle physics to the gravitational case. This text is not an exhaustive review of this rapidly developing field, we refer to the following reviews for the effective field theory methods for the post-Newtonian expansion [52,60], the world-line formalism[5] applied to classical gravity [56, 58], scattering amplitudes for post-Minkowskian expansion [55, 62–66] and effective field theory for quantum gravity [18, 19, 67].

[5]It has been shown [61] that the world-line formalism is equivalent to the scattering amplitude we will not discuss this further.

2 Einstein gravity as an effective field theory

In this text we present the approach consisting in embedding the classical Einstein theory of gravity into a quantum gravity framework where the Einstein-Hilbert action (1) is only the first terms of a low-energy effective action

$$\mathcal{S}_{\text{eff}} = \int d^4x\sqrt{-g}\left[\frac{R}{16\pi G_N} + g^{\mu\nu}T^{\text{matter}}_{\mu\nu} + \mathcal{L}_{\text{corrections}}\right]. \tag{5}$$

The contributions $\mathcal{L}_{\text{corrections}}$ arise from extension of Einstein gravity induced either by high-energy quantum corrections from a high-energy completion like string theory or new interactions from extra massless fields yet undetected.

This approach is based on the idea that long range interactions in gravity can be described by a low-energy effective field theory, even if the high-energy behaviour of quantum gravity is still unknown [18, 19, 67]. Although the status of the high-energy behaviour of quantum gravity is still open, considering effective field theory of gravity at low energy does not pose a problem. One can safely extract low-energy physics from the quantization of the gravitational interactions observables that are independent of the high-energy behaviour. We quote J. D. Bjorken in [68] who argues for this approach

> I also question the assertion that we presently have no quantum field theory of gravitation. It is true that there is no closed, internally consistent theory of quantum gravity valid at all distance scales, But such theories are hard to come by, and in any case, are not very relevant in practice. But as an open theory, quantum gravity is arguably our best quantum field theory, not the worst. Feynman rules for interaction of spin-two gravitons have been written down, and the tree-diagrams (no closed loops) provide an accurate description of physical phenomena at all distance scales between cosmological scales, down to near the Planck scale of 10^{-33} cm. The divergent loop diagrams can be renormalized at the expense of an in-principle infinite number of counterterms appended to the Einstein-Hilbert action. However, their effects are demonstrably small until one probes phenomena at the Planck scale of distances and energies
>
> One way of characterizing the success of a theory is in terms of bandwidth, defined as the number of powers of ten over which the theory is credible to a majority of theorists (not necessarily the same as the domain over which the theory has been experimentally tested). From this viewpoint, quantum gravity, when treated—as described above—as an effective field theory, has the largest bandwidth; it is credible over 60 orders of magnitude, from the cosmological to the Planck scale of distances.

We will assume that the effective field theory satisfies the standard requirements of locality, unitarity and Lorentz invariance, and of course that the theory is diffeomorphism invariant (i.e. we have the symmetries of General relativity). The low-energy degrees of freedom are the massless graviton (from the Einstein-Hilbert term in (5)) and the usual massive matter fields coupled minimally as in (5). Since we will only be interested in the long distance regime (the infrared regime) we will throw away any contributions that diverges at high-energy. That will allow use to quantum gravity techniques at loop order extending the computational power advertised by Bjorken to the loop expansion.

One important application of this idea is that the classical gravity contributions to the two-body interaction, like the classical post-Minkowskian expansion needed for the inspiral phase of gravitational-wave physics, are only sensitive to low-energy degrees of freedom, as long as the energy transfer is small compared to the classical scale of the problem set by the masses of binary system.

We can then use the formalism of quantum field theory and capitalize on the methods developed in particle physics for analytically computing the post-Minkowskian expansion of binary systems. Traditional perturbative (off-shell) quantum field theory calculations in gravity are far from optimal computation-wise. Direct computation from Feynman diagrams is notoriously complex and tiresome due to the complication from tensor calculus. Stimulated by early examples of gravity scattering amplitude computations [69–71] and [72, 73], variety of dedicated approaches [52,57,58,62,64–66,74,75] have been developed and this has been a catalyst for new ideas in connecting Einstein gravity to quantum scattering amplitudes.

In the rest of this text we will explain how one can derive various classical gravity observables realizing the suggestion by Kovàcs and Thorne in [20] to use quantum theory for deriving

- The relativistic two-body potential in Section 3
- Black-hole metrics in Section 4
- Scattering angle in Section 5
- Long-range quantum corrections from the effective field theory in Section 6

One advantage of this approach is that the framework that will be presented in the rest of this text is not restricted to four dimensional physics, but it can be applied to gravity in higher dimensional space-time. This could be important as a step to understand why our four dimensional world is special. As well, the formalism can include any modification of Einstein gravity as long as these modifications are compatible with the locality and unitarity of field theory and respect diffeomorphism invariance.

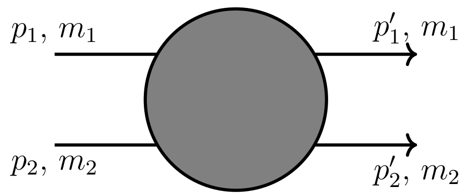
We will therefore present an effective field theory point of view from the compact binary system dynamics. We will show how this framework can lead to accurate theoretical gravitational waveform templates and by using effective field theories of gravity, we can use gravitational wave observations to study gravity in both the weak and strong coupling regimes, and to search for evidence of new physics beyond general relativity.

We will model black-hole gravitational scattering by massive point particles. For the case without spin, see, for instance [76–90]. Current state of the art for such computations is [91–93] reaching fourth post-Minkowskian order while the fifth post-Minkowskian order approached in the probe limit [94]. While our principal emphasis here will be the non-spinning amplitude-based computations, there are also significant improvements for spinning black holes, see, *e.g.*, [95–113] and some based on the world-line approaches, e.g., [113–118].

3 Classical gravity from quantum gravity

We explain how classical physics emerges from quantum scattering in the regime when the energy transfer is small compared to the mass or the energy of the source.

We are interested in extracting physical observables from the gravitational interactions,[6] between two massive body of masses m_i and spin S_i with $i = 1, 2$ interacting via the exchange of massless spin-2 graviton [14–17, 121]



The classical scattering matrix $\mathcal{M}(p_1, p_2, p'_1, p'_2)$ depends on the relativistic factor energy $\gamma := p_1 \cdot p_2/(m_1 m_2) = 1/\sqrt{1-(\vec{v}/c)^2}$ where v is the relative velocity between the two massive objects, the momentum transfer $(p_1 - p'_1)^2 =: \underline{\vec{q}}^2$. At a given order in perturbation one gets the exchange of gravitons (curly lines) between massive external matters (solid lines)

In the case of absence of radiation in the final state the two-body scattering matrix can be expanded in perturbation

$$\mathcal{M}(p_1, p_2, p'_1, p'_2) = \sum_{L=0}^{\infty} G_N^{L+1} \mathcal{M}_L(\gamma, q^2). \tag{6}$$

where $\mathcal{M}_L(\gamma, q^2)$ is the Lth post-Minkowskian contribution at the order G_N^{L+1} in Newton's constant, and it has the polynomial mass dependence

$$\mathcal{M}_L(\gamma, q^2) = \frac{m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma) m_1^{L-i} m_2^i. \tag{7}$$

The classical result is finite in $D = 4$, but at intermediate state of the computation infrared pole appear, but they do not contribute to the observables. In the case of radiation as will be discussed later the matrix elements of the observables still have a polynomial dependence, but the expression has a different expansion in the masses, and it is not only a function of the relativistic factor γ due to asymmetry induced by the radiation on the external lines.

The Klein-Gordon equation for the propagation of a massive scalar field reads

$$\left(\Box - \left(\frac{mc}{\hbar}\right)^2\right) \varphi(t, \vec{x}) = 0, \tag{8}$$

where $\Box = \partial_t^2 - \sum_{i=1}^3 \partial_{x_i}^2$ is the D'Alembertien therefore in quantum mechanics the mass of the field appears in the combination of its inverse Compton wave-length $mc/\hbar$. As well the momenta entering the quantum scattering are the wave-numbers $q = \hbar \underline{q}$.

[6] One could as well include electro-magnetic interactions as considered in [119] or standard model contributions as in e.g. [120] but here we will only consider the exchange of the graviton and focus on the gravitational sector. We will return to the effect of extra massless fields in Section 6.

A traditional argument (see for instance [122]) gives that the L-loop contribution is of order $\mathcal{M}_L(\gamma, q^2) = \mathcal{O}(\hbar^{L-1})$. A different behaviour emerges when keeping fixed the classical momentum $\underline{q} = q/\hbar$ and taking both the $\hbar \to 0$ and the small momentum transfer $q \to 0$ limit [74, 123, 124]. The L-loop two-body scattering amplitude has the Laurent expansion around four dimensions

$$\mathcal{M}_L(\gamma, \underline{q}, \hbar) = \frac{\mathcal{M}_L^{(-L-1)}(\gamma, D)}{\hbar^{L+1} |\underline{q}|^{\frac{L(4-D)}{2}+2}} + \cdots + \frac{\mathcal{M}_L^{(-1)}(\gamma, D)}{\hbar |\underline{q}|^{\frac{L(4-D)}{2}+2-L}} + O(\hbar^0). \tag{9}$$

The full quantum amplitude contains three types of contributions:

1) the term of order $1/\hbar^r$ with $L+2 \leq r \leq 2$ that are more singular than the classical piece in the $\hbar \to 0$ limit.

2) the classical piece of order $1/\hbar$ from which the classical Einstein gravity contribution is extracted. It showed the presence of a classical piece in the quantum gravitational two-body amplitude at one-loop [69], as articulated as an all order statement in [123]. The expansion in (9) is unusual, but this is a natural when considering a huge external mass expansion of the two-body gravitational scattering. At the $L+1$ post-Minkowskian order, the two-body scattering amplitude between two massive particles have the following mass dependence given in (7). This classical contribution emerges from the $1/\hbar$ piece of the quantum amplitude in (9) remembering that the mass dependence in quantum field theory appears as the Compton wave-length $mc/\hbar$. Expressing the classical contribution by making this explicit gives

$$\mathcal{M}_L(\gamma, q^2, \hbar) = \cdots + \underbrace{\frac{m_1^2 m_2^2}{\underline{q}^{2+\frac{(2-D)L}{2}}} \hbar^{L-1} G_N^{L+1} \sum_i \left(\frac{m_1 c}{\hbar}\right)^{L-i} \left(\frac{m_2 c}{\hbar}\right)^i}_{=\frac{\mathcal{M}_L(\gamma, \underline{q}^2)}{\hbar}} + \cdots \tag{10}$$

Therefore the polynomial mass dependence of the classical amplitude in (7) expected for the conservative part of the scattering angle [78, 125] arises consistently from the classical limit of the quantum amplitude. The $\underline{q}^2$ dependence of the classical contributions are exactly what one anticipates to contribute to the three-dimensional potential at the $L+1$ post-Newtonian order since

$$G_N^{L+1} \int d^3\vec{\underline{q}} \frac{e^{i\underline{q}\cdot\vec{r}}}{|\underline{q}|^{2-L}} \propto \left(\frac{G_N}{r}\right)^{L+1}. \tag{11}$$

The expansion in (9) indicates that a given $\hbar$ order has a prescribed analytic dependence in q^2. We utilize this to elucidate the classical contribution from specific unitarity cuts [74]. The extraction of the classical part has been since systematized using a heavy-mass effective theory approach [90, 126], or the velocity cut formalism [87, 88, 94].

3) the quantum corrections of order $\hbar^r$ (with $r \geq 0$) leads to quantum gravity corrections to the classical Einstein gravity results [127].

All these contributions are constrained by the unitarity of the S-matrix as we will explain in Section 5.3. We illustrate in sections 3.1 and 3.2 how the small $\hbar \to 0$

expansion arises from tree-level and one-loop amplitude and how the classical piece of the scattering amplitude can be simply recovered.

There is a systematic procedure for connecting the classical contribution of order $\hbar^{-1}$ of the scattering amplitude in perturbative gravity with post-Minkowskian potentials in classical General Relativity.

Assuming the existence of a relativistic one-particle Hamiltonian of only particle states describing what in bound-state problems is known as the Salpeter equation,

$$\mathcal{H}_{\rm PM}(r,p) = \sqrt{p^2+m_1^2} + \sqrt{p^2+m_2^2} + V(p,r) \tag{12}$$

with in the centre-of-mass frame the momenta of the scattering particle are parametrized as $p_1 = (E_1, \vec{p})$, $p_2 = (E_2, \vec{p}')$, $p_1' = (E_1, -\vec{p})$ and $p_2' = (E_2, -\vec{p}')$ with $|\vec{p}| = |\vec{p}'|$.

The relativistic potential $V(p,r)$ has an expansion in the powers of Newton's constant $V(p,r) = \sum_{n\geq 1} G_N^n V_{n-\rm PM}(p,r)$ where the nth post-Minkowskian potential sums an infinite number of post-Newtonian contributions in the small velocity expansion $v/c \ll 1$, as $V_{\rm L+1-PM}(p,r) = \sum_{n\geq 0} c_{L,n}(r) v^n$, with $p = p_1 - p_1' = (E, v)$.

The position space potential is obtained by

$$V(p,r) = \int \frac{d^3q}{(2\pi)^3} e^{iq\cdot r} \langle p|V|p'\rangle \tag{13}$$

with the Lippmann-Schwinger equation connecting the scattering matrix elements to the potential

$$\langle p|\hat{T}(z)|p'\rangle = \langle p|\hat{V}|p'\rangle + \int \frac{d^3k}{(2\pi)^3} \frac{\langle p|\hat{V}|k\rangle\langle k|\hat{T}(z)|p'\rangle}{z - E_k} \tag{14}$$

and use the crucial relation

$$\lim_{\epsilon\to 0} \langle p|\hat{T}(E_p + i\epsilon)|p'\rangle = \mathcal{M}(p,p') \tag{15}$$

which provides the link to the conventionally defined scattering amplitude $\mathcal{M}$ in quantum field theory restricted to the particle sector.

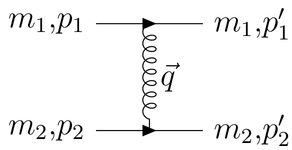
The scattering amplitude approach completes the post-Newtonian computations by providing information beyond its regime of validity and leads to surprising results connecting the conservative part and gravitational radiation effects [73, 81, 83–87, 128]. It gives a new perspective on the traditional methods [52, 53, 57, 129, 130] used for computing the gravitational-wave templates. This approach allows connecting the re-summed post-Newtonian results [77,79,91] and the high-energy behaviour [82,85]. We can then explore the behaviour of the post-Minkowskian expansion for higher-dimensional gravity. One can as well consider higher derivative corrections induced from string theory for instance, and study their effects on the gravitational-wave signals [131–133] or various quantum effects [127] induced from the higher-order terms in the $\hbar$ expansion in (9).

3.1 The relativistic potential at the first Post-Minkowskian order

We illustrate the emergence of the classical and quantum pieces at tree-level order.

The tree-level scattering between two massive fields has the following $\hbar$ expansion

$$\mathcal{M}_0(\gamma, \underline{q}^2, \hbar) =$$



$$= -\frac{16\pi G_N m_1^2 m_2^2 \,(2\gamma^2 - 1)}{\hbar |\underline{q}|^2} + \hbar 4\pi G_N p_1 \cdot p_2. \tag{16}$$

In this expression, the contribution $G_N m_1^2 m_2^2 (2\gamma^2 - 1)$ is the classical first post-Minkowskian matching the results from general relativity see e.g. [128], and the higher order quantum correction $\hbar p_1 \cdot p_2$ due to the contact term as mandated by the full quantum amplitude. This corresponds to short distance interaction. Since we are only interested into the long range effect these contributions are negligible.

Taking the three-dimensional Fourier transform using

$$\int e^{i\vec{r}\cdot\vec{q}} |q|^\alpha \frac{d^D\vec{q}}{(2\pi)^D} = \frac{(2\pi)^{1-D}}{2} \frac{\Gamma\left(\frac{\alpha+D}{2}\right)}{\Gamma\left(\frac{2-\alpha-D}{2}\right)} \left(\frac{2}{r}\right)^{\alpha+D} \tag{17}$$

From this we deduce the relativistic invariant potential at the first post-Minkowskian order

$$V_{\rm 1PM}(p, r) = \frac{1}{E_1 E_2} \frac{G_N m_1^2 m_2^2 (1 - 2\gamma^2)}{r} \tag{18}$$

where E_i is the energy of the particle i.

3.2 The relativistic potential at the second Post-Minkowskian order

In reference [71], we elucidated that generalized unitarity is an excellent tool to calculate terms that resemble long-range contributions in amplitudes. Such non-analytic terms provide us with classical scattering potentials in theories such as QED, gravity, and quantum modifications. Since we are exclusively interested in non-polynomial contributions, we are not required to generate the full amplitude. Identifying those terms in the amplitude is adequate for classical and leading quantum corrections. Thus, a pathway is established to streamline such computations. At one-loop order, we fetch coefficients corresponding to $1/\sqrt{-q^2}$ and $\log(-q^2)$ terms in the amplitude from on-shell unitarity. Following the approach provisioned in [134], this can, *e.g.* be done through evaluating the phase-space integrals by reinstating the off-shell cut propagators with on-shell cut conditions in numerators.

Formally at one loop, we thus have to consider the cut associated with the integral expression

$$iM^{1-\rm loop}\big|_{disc} = \int \frac{d^D\ell}{(2\pi)^D} \frac{\sum_{\lambda_1,\lambda_2} M^{\rm tree}_{\lambda_1\lambda_2}(p_1, p_1', -\ell_2^{\lambda_2}, \ell_1^{\lambda_1})(M^{\rm tree}_{\lambda_1\lambda_2}(p_2, p_2', \ell_2^{\lambda_2}, -\ell_1^{\lambda_1}))^*}{\ell_1^2 \ell_2^2}\Big|_{cut}, \tag{19}$$

Here we have the cut conditions $\ell_1^2 = \ell_2^2 = 0$, and we sum over all feasible physical graviton helicity arrangements across the cut: λ_1 and λ_2. Box, triangle, and bubble graph topologies supply the basis for the quantum and classical contributions at one loop. In the cut, we can directly pinpoint the integral functions and thus isolate the coefficients for the non-analytic terms we are pursuing. In [88], we separated the classical non-analytically contributions by evaluating the triple cut and recognizing the coefficients for the two-loop basis of integral functions. We expand at one-loop order, the full quantum two-body scattering amplitude on the standard basis of scalar one-loop integrals in four dimensions

$$\mathcal{M}_1 = \frac{i16\pi^2 G_N^2 m_1^2 m_2^2}{E_1 E_2}\left(3\left(1-5\gamma^2\right)\left(m_1^2 I_{\triangleright} + m_2^2 I_{\triangleleft}\right) + 4m_1^2 m_2^2\left(1-2\gamma^2\right)^2\left(I_{\square} + I_{\bowtie}\right) + \cdots\right), \tag{20}$$

where $E_i^2 = \vec{p}_i^2 + m_i^2$ with $i = 1, 2$ is the energy of the particle $i = 1, 2$. The expression contains the massive scalar triangles, which have the large mass expansion exhibiting the classical $1/\hbar$ term [69, 74, 123]

$$I_{\triangleright} = -\frac{i}{32m_1}\frac{1}{\hbar|\underline{q}\,|} + \cdots, \qquad I_{\triangleleft} = -\frac{i}{32m_2}\frac{1}{\hbar|\underline{q}\,|} + \cdots \tag{21}$$

and the scalar box and cross-box integrals

$$\begin{aligned} I_{\square} &= \frac{i}{16\pi^2\hbar^2|\underline{q}\,|^2}\left(-\frac{1}{m_1 m_2} + \frac{m_1(m_1 - m_2)}{3m_1^2 m_2^2} + \frac{i\pi}{|p|(E_1 + E_2)}\right)\left(\frac{2}{3-d} - \hbar^2 \log|q|^2\right) + \cdots \\ I_{\bowtie} &= \frac{i}{16\pi^2\hbar^2|\underline{q}\,|^2}\left(\frac{1}{m_1 m_2} - \frac{m_1(m_1 - m_2)}{3m_1^2 m_2^2}\right)\left(\frac{2}{3-d} - \hbar^2 \log|q|^2\right) + \cdots \end{aligned} \tag{22}$$

putting everything together, we get that expansion of the total quantum one-loop amplitude read

$$\begin{aligned} \mathcal{M}_1(\gamma, \underline{q}^2, \hbar) = \frac{\pi^2 G_N^2 m_1^2 m_2^2}{E_1 E_2}\Bigg[&-\frac{3\left(1-5\gamma^2\right)}{2\hbar|\underline{q}\,|}(m_1 + m_2) \\ &+ \frac{i m_1 m_2}{(E_1 + E_2)}\frac{4\left(1-2\gamma^2\right)^2}{|\vec{p}\,|}\frac{\left(\frac{2}{3-d} - \hbar^2 \log|q|^2\right)}{\pi\hbar^2|\underline{q}|^2}\Bigg] + \cdots. \end{aligned} \tag{23}$$

This expression contains

- at order $1/\hbar^2$ a contribution given by the square of the classical tree-level contribution from (16). This piece we need for the exponentiation of the S-matrix in (56) as detailed in [89].
- At order $1/\hbar$ the classical second post-Minkowskian contribution [74] that matches the classical second post-Minkowskian result for generic masses.
- A quantum piece of order $\hbar^0$ which is a long-range infrared quantum gravity effect. Because this is the first quantum correction to the classical result, the value of the quantum gravity-induced correction is universal and independent of the ultraviolet regularization [71, 127].

In order to consider a post-Minkowskian potential at second order in G_N^2, we will need to consider a contribution coming from the iterated tree-level amplitude, as dictated by (14)

$$V_{2\mathrm{PM}}(p, q) = \mathcal{M}^{1-\mathrm{loop}}(p, p') + \mathcal{M}^{\mathrm{Iterated}}(p, p') \tag{24}$$

$$\mathcal{M}^{\text{Iterated}}(p,p') \equiv -\int \frac{d^dk}{(2\pi)^d} \frac{\mathcal{M}^{\text{tree}}(p,k)\mathcal{M}^{\text{tree}}(k,p')}{E_p - E_k + i\epsilon} \,. \tag{25}$$

The imaginary part of this which arises from the box and crossed-box integrals is the infrared divergent Weinberg phase [135]. By restoring the $\hbar$-counting, one sees that it scales as $\hbar^{-1}$, a behaviour dubbed super-classical in [124]. We will show below that it cancels in the properly defined potential, a fact already noted in the post-Newtonian expansion [136].

$$\begin{aligned}\mathcal{M}^{\text{Iterated}} = &\frac{i\pi G_N^2}{(E_1+E_2)E_1E_2} \frac{4c_1^2}{|\vec p|} \frac{(\log|\vec q|^2 - \frac{2}{3-d})}{|\vec q|^2} \\ &+ \frac{2\pi^2 G_N^2(E_1+E_2)}{E_1^2E_2^2|\vec q|}\left(\frac{c_1^2(E_1E_2+E_1^2+E_2^2)}{2E_1E_2(E_1+E_2)^2} - 4c_1 p_1\cdot p_3\right)\end{aligned} \tag{26}$$

The second-order post-Minkowskian potential in momentum space is thus given by

$$V_{2\text{PM}}(p,q) = \mathcal{M}^{1-\text{loop}} + \mathcal{M}^{\text{Iterated}} \tag{27}$$

leading to

$$V_{2\text{PM}}(p,q\,) = \frac{\pi^2 G_N^2}{E_1E_2|\vec q\,|}\left[\frac{1}{2}\left(\frac{c_\triangleright}{m_a}+\frac{c_\triangleleft}{m_b}\right)+\frac{2(E_1+E_2)}{E_1E_2}\left(\frac{c_1^2(E_1E_2+E_1^2+E_2^2)}{2(E_1+E_2)^2E_1E_2}-4c_1p_1\cdot p_3\right)\right] \tag{28}$$

or, in coordinate space,

$$V_{2\text{PM}}(p,r) = \frac{G_N^2}{r^2}\frac{1}{E_1E_2}\left[\frac{1}{4}\left(\frac{c_\triangleright}{m_a}+\frac{c_\triangleleft}{m_b}\right)+\frac{(E_1+E_2)}{E_1E_2}\left(\frac{c_1^2(E_1E_2+E_1^2+E_2^2)}{2(E_1+E_2)^2E_1E_2}-4c_1p_1\cdot p_3\right)\right]. \tag{29}$$

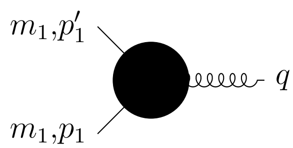
This agrees with what has been previously obtained in ref. [75]. As expected on physical grounds, the imaginary part which is composed of super-classical and infrared divergent pieces has cancelled, leaving a finite and well-defined post-Minkowskian potential at $d=3$. That such cancellation had to occur was expected on physical ground, since the imaginary part clearly cannot affect classical motion.

4 Classical black holes metrics

Black-hole solutions are a perfect play ground to validate the formalism of deriving classical gravity from quantum scattering amplitudes. This also opens new avenues for studying black holes in generalized theories of gravity.

In 1973 Duff analysed [137] the question of the classical limit of quantum gravity by extracting the Schwarzschild back hole metric from quantum tree graphs to G_N^3 order. This was a consistency check on the way classical Einstein's gravity is embedded into the standard massless spin-2 quantization of the gravitational interactions.

By evaluating the vertex function of the emission of a graviton from a particle of mass m, spin S and charge Q, in d dimensions



$$= -\frac{i\sqrt{32\pi G_N}}{2}\sum_{l\geq 0}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu} \tag{30}$$

with the action

$$\mathcal{S} = \int d^{d+1}x\sqrt{-g}\left(\frac{R}{16\pi G_N} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{31}$$

one can extract the metric of physical black holes [138]

- Schwarzschild black hole: Scalar field $S = 0$, mass m [56, 70, 139, 140]
- Reissner-Nordström black hole: Scalar field $S = 0$, charge Q, mass m [119]
- Kerr-Newman black hole: Fermionic field $S = \frac{1}{2}$, charge Q, mass m [119, 139]

At each loop order we extract the l-loop contribution to the transition density of the stress-energy tensor $\langle T_{\mu\nu}(q^2)\rangle = \sum_{l\geq 0}\langle T^{(l)}_{\mu\nu}(q^2)\rangle$

$$i\mathcal{M}_3^{(l)}(p_1, q) = -\frac{i\sqrt{32\pi G_N}}{2}\langle T^{(l)\,\mu\nu}(q^2)\rangle\epsilon_{\mu\nu}, \tag{32}$$

where $\epsilon^{\mu\nu}$ is the polarization of the graviton with momentum $q = p_1 - p_2$ is the momentum transfer.

The scattering amplitude computation is not done in the harmonic gauge coordinates $g^{\mu\nu}\Gamma^\lambda_{\mu\nu}(g) = 0$, but in the *de Donder gauge* coordinate system [17, 56, 141–143]

$$\eta^{\mu\nu}\Gamma^\lambda_{\mu\nu}(g) = \eta^{\mu\nu}g^{\lambda\rho}\left(\frac{\partial g_{\rho\mu}}{\partial x^\nu} + \frac{\partial g_{\rho\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\rho}\right) = 0 \tag{33}$$

the metric perturbations $g_{\mu\nu} = \eta_{\mu\nu} + \sum_{n\geq 1} h^{(n)}_{\mu\nu}$ satisfy[7]

$$\frac{\partial}{\partial x^\lambda}h^{\lambda(n)}_\nu - \frac{1}{2}\frac{\partial}{\partial x^\nu}h^{(n)} = 0\,. \tag{34}$$

The de Donder gauge relation between the metric perturbation and the stress-energy tensor reads

$$h^{(l+1)}_{\mu\nu}(\vec{x}) = -16\pi G_N\int\frac{d^d\vec{q}}{(2\pi)^d}e^{i\vec{q}\cdot\vec{x}}\frac{1}{\vec{q}^2}\left(\langle T^{(l)}_{\mu\nu}\rangle^{\text{class.}}(q^2) - \frac{1}{d-1}\eta_{\mu\nu}\langle T^{(l)}\rangle^{\text{class.}}(q^2)\right). \tag{35}$$

In this relation enters the classical contribution at l loop order $\langle T^{(l)}_{\mu\nu}\rangle^{\text{class.}}(q^2)$ defined by the classical limit of the quantum scattering amplitude [74, 123, 124] and its application to black hole solutions [102].

In [140], the Schwarzschild metric up to G_N^4 obtained in four ($d = 3$), five ($d = 4$) and six ($d = 5$) dimensions. The Schwarzschild-Tangherlini metric in the de Donder coordinate system

$$\begin{aligned} ds^2 = &\left(1 - 4\frac{d-2}{d-1}\frac{\rho(r,d)}{f(r)^{d-2}}\right)dt^2 - f(r)^2 d\vec{x}^2 \\ &- \left(-f(r)^2 - f(r)^{d-2}\frac{(f(r) + r\frac{df(r)}{dr})^2}{f(r)^{d-2} - 4\frac{d-2}{d-1}\rho(r,d)}\right)\frac{(\vec{x}\cdot d\vec{x})^2}{\vec{x}^2} \end{aligned} \tag{36}$$

[7] The harmonic gauge linearized at the first order in perturbation gives (34) with $n = 1$. The higher-order expansions of the harmonic gauge differ from these conditions.

and to the first order the components of the metric are given [140] In four dimensions $(d=3)$[8]

$$h_0^{\rm dD}(r) = 1 - \frac{2G_Nm}{r} + 2\left(\frac{G_Nm}{r}\right)^2 + 2\left(\frac{G_Nm}{r}\right)^3 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_Nm}\right) - 6\right)\left(\frac{G_Nm}{r}\right)^4 + \cdots \tag{37}$$

$$\begin{aligned}h_1^{\rm dD}(r) = 1 + 2\frac{G_Nm}{r} + 5\left(\frac{G_Nm}{r}\right)^2 + \left(\frac{4}{3}\log\left(\frac{rC_3}{G_Nm}\right) + 4\right)\left(\frac{G_Nm}{r}\right)^3 \\ + \left(-\frac{4}{3}\log\left(\frac{rC_3}{G_Nm}\right) + \frac{16}{3}\right)\left(\frac{G_Nm}{r}\right)^4 + \left(\frac{64}{15}\log\left(\frac{rC_3}{G_Nm}\right) - \frac{26}{75}\right)\left(\frac{G_Nm}{r}\right)^5 \\ + \left(\frac{4}{9}\log\left(\frac{rC_3}{G_Nm}\right)^2 - \frac{24}{5}\log\left(\frac{rC_3}{G_Nm}\right) + \frac{298}{75}\right)\left(\frac{G_Nm}{r}\right)^6 + \cdots\end{aligned}$$

One notices that in this gauge the metric components contains finite-size powers of $\log(rC_3/G_Nm)$ where C_3 is the single constant of integration. These logarithms are generated by the cancellation of the ultraviolet divergences of the scattering amplitudes in (32) regulated by the introduction of higher-derivative non-minimal couplings [56,140,143]. These contributions are finite size effects [52,56,59,133,144–147]. At the level of the metric components, they are reabsorbed in the coordinate change from the de Donder gauge used for the amplitude computation to the standard Schwarzschild-Tangherlini metric in spherical coordinates

$$ds^2 = \left(1 - 4\frac{d-2}{d-1}\rho(r,d)\right)dt^2 - d\vec{x}^2 - \frac{4\frac{d-2}{d-1}\rho(r,d)}{1 - 4\frac{d-2}{d-1}\rho(r,d)}\frac{(\vec{x}\cdot d\vec{x})^2}{r^2} \tag{38}$$

with

$$\rho(r,d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{d-2}{2}}}\frac{G_Nm}{r^{d-2}}\,, \tag{39}$$

and the finite-size effects do not affect the static metric.

Several amplitudes based methods have been developed for deriving black hole metrics, in particular the Kerr metric [63]. Many black holes involved in the production of gravitational waves are Kerr black, and the inclusion of the angular momentum (classical spin) of the black hole is important [62].

5 Classical scattering angle

The scattering angle is an essential observable for connecting the scattering regime presented in the previous section with the bound-state region describing the inspiral phase of the binary system. The scattering angle is the main link for connecting the scattering amplitudes to the dynamics of the two-body system. It has been derived in many independent methods in different regimes [79, 81, 84–86, 114, 118, 148, 149].

We start by discussing the scattering angle from classical general relativity for the conservative sector, and we then show how to extend it to include radiation.

5.1 Radial action

In the conservative sector the total energy E given by the Hamiltonian (12) and angular momentum J are conserved quantities so that the total momentum in the

[8] Similar results are obtained in higher dimensions where we matched the Schwarzschild-Tangherlini up to the order G_N^4.

centre-of-mass can be decomposed as $p(r,E)^2 = p_r^2(r,J,E) - J^2/r^2$. The Hamilton-Jacobi equations applied to the principal function

$$\mathcal{S}(r,\varphi;J,E) = J\varphi + \int p_r(r;J,E)dr \tag{40}$$

leads to the radial action [47, 72, 150]

$$\mathcal{S}_{\text{radial}}(J,E) = \oint p_r(r;E,J)dr = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{p^2(r,E) - \frac{J^2}{r^2}}dr \tag{41}$$

where the integration is between the root of $p^2(r,E) - \frac{J^2}{r^2} = 0$ with the convention that $0 < r_- < r_+$. so that the periastron advance is obtained in the bound state problem $E < 0$ by

$$\frac{\Phi}{2\pi} = -\frac{\partial \mathcal{S}_{\text{radial}}}{\partial J} \tag{42}$$

and the scattering angle in the scattering regime $E > 0$ by a similar formula. The two regimes are connected by analytic continuation as discussed in [148, 149] in the context of gravitational interaction.

For the Newtonian potential $V_{\text{Newton}} = -G_N m_1 m_2/r$ the scattering angle takes a closed form expression

$$\tan\left(\frac{\chi_{\text{Newton}}(J,E)}{2}\right) = \frac{G_N m_1 m_2}{J}\sqrt{\frac{M}{E-M}}\sqrt{\frac{m_1 m_2}{2(m_1+m_2)^2}}. \tag{43}$$

In the post-Minkowskian expansion no closed form formula is known, but the scattering angle can be expanded powers of G_N [72, 128, 148, 149, 151, 152]

$$\frac{\chi_{\text{PM}}(J,E)}{2} = \sum_{n\geq 1}\chi^{(n)}(E)\left(\frac{G_N m_1 m_2}{J}\right)^n. \tag{44}$$

Once we have reconstructed the post-Minkowskian Hamiltonian, along the lines presented above, we can compute the scattering angle in perturbation in G_N. Because at least up to and including third post-Minkowskian order, there exists, in isotropic coordinates, a very simple relationship between centre-of-mass momentum p and the effective classical potential $V(r,p)$ of the form [152] and [149]

$$p(r,E)^2 = p_\infty^2 - V(r,E); \qquad V(r,E) = -\sum_{n\geq 1} f_n \left(\frac{G_N(m_1+m_2)}{r}\right)^n \tag{45}$$

where the coefficients f_n are directly extracted from the scattering angle

$$\chi_{\text{PM}}(J,E) = \sum_{k\geq 1}\frac{2J}{k!}\int_0^\infty du \left(\frac{d}{du^2}\right)^k \left[\frac{1}{u^2+J^2}\left(\frac{V\left(\sqrt{J^2+b^2}/p_\infty^2, E\right)(u^2+J^2)}{\gamma^2-1}\right)^k\right]. \tag{46}$$

By computing the two-body scattering in perturbation one derives a Lorentz invariant expression valid in all regime of relative velocity between the two interacting massive bodies. One route to connect the scattering regime to the bound-state regime is based on the Effective One-Body (EOB) formalism [153, 154], suitably adapted from post-Newtonian to post-Minkowskian formulations [72, 73, 128, 155]. Importantly, the relation between the scattering amplitude and the Effective-One-Body Effective potential in (46) is valid in any space-time dimension and applies to gravity in higher dimensions [156, 157]. Something that we will comment further in Section 6.

5.2 The eikonal formalism

It has been proposed in [91] a relation between the scattering amplitude and the radial action (see [158] for the probe limit case)

$$i\mathcal{M} \propto \int \left(e^{iI_{\rm radial}(E,J)} - 1\right) dJ. \tag{47}$$

This relation looks similar to the eikonal approach.

The eikonal method is a technique in quantum field theory for relating the scattering amplitude to the scattering angle [55,82,85,127,159]. For this, one converts the amplitude to the b-space[9] by performing a Fourier transform with respect to the momentum transfer

$$\mathcal{M}_L(\gamma, b) = \frac{1}{4m_1m_2\sqrt{\gamma^2-1}} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2}\underline{\vec q}}{(2\pi)^{D-2}} \mathcal{M}_L(p_1,p_2,p_1',p_2') e^{i\underline{\vec q}\cdot\vec b}. \tag{48}$$

The classical eikonal phase $\delta(\gamma, b)$ is defined by the exponentiation of the S-matrix

$$1 + i\mathcal{T} = (1 + i2\Delta) e^{\frac{2i\delta(\gamma,b)}{\hbar}}. \tag{49}$$

The eikonal phase has the perturbation expansion

$$\delta(\gamma, b) = \sum_{L\geq 0} \delta_L(\gamma, b) G_N^{L+1}, \tag{50}$$

which is then connected to the $\hbar$ Laurent expansion of the scattering amplitude in (9), through the expansion of the full scattering matrix in b-space

$$1 + i\mathcal{T} = 1 + i \sum_{L\geq 0} G_N^{L+1} \mathcal{M}_L(\gamma, b). \tag{51}$$

Having determined the classical eikonal contribution at a given loop order one can then evaluate the scattering angle at this order in perturbation [81, 85, 87, 88]

$$\sin\left(\frac{\chi}{2}\right)\Big|_L = -\frac{\sqrt{(p_1+p_2)^2}}{m_1m_2\sqrt{\gamma^2-1}} \frac{\partial \delta_L(\gamma, b)}{\partial b}. \tag{52}$$

By plugging the perturbative expansion for the angle in (44) one recovers the link between the scattering amplitude and the angle described above.

This link between the eikonal phase and the scattering amplitude explains why the small $\hbar$ expansion of the amplitude takes the form given in (9).

[9] This is not the impact parameter b_J orthogonal to the asymptotic momentum in the centre-of-mass frame. The relation between the two quantities is $b_J = b\cos(\chi/2)$ [81, 85].

The expansion of the exponentiation formula in (49) gives the double series expansion in $\hbar$ and G_N

$$e^{\frac{2i\delta(\gamma,b)}{\hbar}} = \sum_{L\geq 0} G_N^{L+1} \sum_{\substack{n_1+2n_2+\cdots=L \\ n_i\geq 0}} \prod_{r\geq 1} \left(\frac{2i\delta_{r-1}(\gamma,b)}{\hbar}\right)^{n_r} \tag{53}$$

which show that each given order in G_N the inverse powers of $\hbar$ in (9) are needed for the exponentiation of the eikonal phase.

Unfortunately, this approach leads to complicated computations. In the first place, the eikonal exponentiation in (49) is obtained after a careful separation, order by order, of the various terms that go into the exponent and those terms that remain as prefactor at the linear level. A second complication is that after exponentiation in impact-parameter space one must apply the inverse transformation and seek from it two crucial ingredients: (1) the correct identification of the transverse momentum transfer $\underline{\vec{q}}$ in the centre-of-mass frame and (2) the correct identification of the scattering angle from the saddle point. At low orders in the eikonal expansion, this procedure works well, but it hinges on the impact-parameter transformation being able to undo the convolution product of the momentum-space representation. When $\underline{q}^2$-corrections are taken into account it is well-known that this procedure requires amendments. This motivates why alternative pathways are rooted in the WKB approximation [89, 91, 158].

5.3 An exponential representation of the S-matrix

Another approach, introduced in [89], uses an exponential representation of the S-matrix at the operator level

$$\widehat{S} = \mathbb{I} + \frac{i}{\hbar}\widehat{T} = \exp\left(\frac{i\widehat{N}}{\hbar}\right), \tag{54}$$

with the completeness relation

$$\mathbb{I} = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{2} \frac{d^{D-1}k_i}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{k_1}} \prod_{j=1}^{n} \frac{d^{D-1}\ell_j}{(2\pi\hbar)^{D-1}} \frac{1}{2E_{\ell_j}} \left|k_1, k_2; \ell_1, \ldots \ell_n\right\rangle \left\langle k_1, k_2; \ell_1, \ldots \ell_n\right|, \tag{55}$$

which includes all the exchange of gravitons for $n \geq 1$ entering the radiation-reaction contributions $\hat{N}^{\rm rad}$. With this exponential representation of the S-matrix, we systematically relate matrix elements of the operator in the exponential $\hat{N}$ to ordinary Born amplitudes minus pieces provided by unitarity cuts [89]. This is seen by the perturbation expansion

$$\begin{aligned}
&\hat{N}_0 = \hat{T}_0, \qquad \hat{N}_0^{\rm rad} = \hat{T}_0^{\rm rad}, \\
&\hat{N}_1 = \hat{T}_1 - \frac{i}{2\hbar}\hat{T}_0^2, \qquad \hat{N}_1^{\rm rad} = \hat{T}_1^{\rm rad} - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_0^{\rm rad} + \hat{T}_0^{\rm rad}\hat{T}_0), \\
&\hat{N}_2 = \hat{T}_2 - \frac{i}{2\hbar}(\hat{T}_0^{\rm rad})^2 - \frac{i}{2\hbar}(\hat{T}_0\hat{T}_1 + \hat{T}_1\hat{T}_0) - \frac{1}{3\hbar^2}\hat{T}_0^3,
\end{aligned} \tag{56}$$

and similarly for higher orders. The simplicity of this method seems very appealing and suggests that it may be used to streamline post-Minkowskian amplitudes in gravity by means of a diagrammatic technique that systematically avoids the evaluation of the cut diagrams that must be subtracted, but simply discards them at the integrand level. This decomposition is in correspondence with the $1/\hbar|\underline{q}|$ expansion of the scattering amplitude in (9). The scattering matrix operator $\hat{T}$ is related to the scattering amplitude $\mathcal{M}_L \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_L|p'_1, p'_2\rangle$. The tree-level matrix element for the two-body scattering $\mathcal{M}_0 \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_0|p'_1, p'_2\rangle$ is of order $\mathcal{O}(1/\hbar)$. At one-loop order amplitude decomposes into two pieces

$$\mathcal{M}_1 \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{T}_1|p'_1, p'_2\rangle \propto \frac{1}{\hbar}\langle p_1, p_2|\hat{N}_1|p'_1, p'_2\rangle + \frac{i}{2\hbar^2}\langle p_1, p_2|\hat{T}_0^2|p'_1, p'_2\rangle. \tag{57}$$

By unitarity the coefficient of the $\mathcal{O}(1/\hbar^2)$ contribution in the scattering amplitude is $\langle p_1, p_2|\hat{T}_0^2|p'_1, p'_2\rangle$, and the matrix element $\langle p_1, p_2|\hat{N}_1|p'_1, p'_2\rangle$ is given by the classical piece is of order $\mathcal{O}(1/\hbar)$. Therefore, for the classical two-body scattering only the matrix elements of $\hat{N}$ are needed. They are extracted from the scattering amplitude by the *velocity cuts* introduced recently [87,88] which are a practical way of realizing the decomposition (57) at the amplitude level. These velocity cuts provide a natural way to organize amplitude calculations [94].

By construction, the scattering angle is reproduced in perturbation theory. The completeness relation implies that the two-body scattering contains the multi-graviton exchanges. Therefore, the result is not limited to what is known as the potential region of the multi-loop amplitudes [77, 79, 91], but include also radiation reaction pieces [82, 84, 85, 87, 88].

5.4 Gravitational observables

We now show how to use the previous quantum scattering formalism for evaluating classical gravitational observables. The KMOC formalism as originally defined in [124] considers an initial *in*-state of two massive scalars at time $t = -\infty$,

$$|\mathrm{in}\rangle = \int d\Pi_{p_1} d\Pi_{p_2} \tilde{\Phi}_1(p_1)\tilde{\Phi}_2(p_2) e^{\frac{i}{\hbar} b p_1}\, |p_1, p_2; 0\rangle \tag{58}$$

where the state $|p_1, p_2; 0\rangle$ is a momentum eigenstate of two massive scalars and the "0" indicates that there is no radiation present at $t = -\infty$. In the classical limit the wave-functions $\tilde{\Phi}(p_i)$ are chosen to represent two localized scalars separated by impact parameter b^μ. A complete set of states containing an arbitrary number of gravitons is as described in (55), but the initial state at $t = -\infty$ is taken to be free of gravitons, as shown.

A change in an observable corresponding to an operator $\hat{O}$ from $t = -\infty$ to $t = +\infty$ is then [124],

$$\langle \Delta \hat{O}\rangle = \langle \mathrm{in}|\, \hat{S}^\dagger \hat{O} \hat{S}\, |\mathrm{in}\rangle - \langle \mathrm{in}|\, \hat{O}\, |\mathrm{in}\rangle = \langle \mathrm{in}|\, \hat{S}^\dagger [\hat{O}, \hat{S}]\, |\mathrm{in}\rangle\ . \tag{59}$$

Using the linear Born representation of the S-matrix (54) leads to the KMOC formula [124]

$$\langle \Delta \hat{O}\rangle = \frac{i}{\hbar}\, \langle \mathrm{in}|\, [\hat{O}, \hat{T}]\, |\mathrm{in}\rangle + \frac{1}{\hbar^2}\, \langle \mathrm{in}|\, \hat{T}^\dagger [\hat{O}, \hat{T}]\, |\mathrm{in}\rangle \tag{60}$$

In the small $\hbar$ limit this expression leads to the evaluation of the change in a classical observable after the delicate cancellations of higher powers of $1/hbar$ in the expansion.

Here we instead explore consequences of using the exponential representation of the S-matrix. This will lead to a simple and efficient way to extract the change in a classical observable, including dissipative effects.

In an alternative viewpoint we consider the change $\Delta\hat{O}$ of an operator $\hat{O}$ from $t = -\infty$ to $t = +\infty$ as

$$\Delta\hat{O} = \hat{S}^\dagger \hat{O} \hat{S} - \hat{O} \ . \tag{61}$$

which then has to be evaluated between *in*-states of $t = -\infty$. Inserting the exponential representation of the $\hat{S}$ operator of eq. (54) together with the crucial property of Hermiticity of $\hat{N}$,

$$\Delta\hat{O} = e^{-\frac{i\hat{N}}{\hbar}} \hat{O} e^{\frac{i\hat{N}}{\hbar}} - \hat{O} \ . \tag{62}$$

allows us to rewrite eq. (62) by means of the Campbell identity that expands the two exponentials as an infinite sum of nested commutators,

$$\Delta\hat{O} = \sum_{n\geq 1} \frac{(-i)^n}{\hbar^n n!} \underbrace{[\hat{N}, [\hat{N}, \ldots, [\hat{N}, \hat{O}]]]}_{\text{n times}} . \tag{63}$$

This rewriting, which is where we use unitarity of the S-matrix, will play a crucial role in our all-order proofs because it displays the iterative structure of the KMOC formalism when combined with the exponential representation. It is convenient to define

$$\hat{A}_n^{\hat{O}} \equiv \frac{1}{\hbar^n} \underbrace{[\hat{N}, [\hat{N}, \ldots, [\hat{N}, \hat{O}]]]}_{\text{n times}} . \tag{64}$$

The nested commutator structure implies the operator relation

$$\hat{A}_n^{\hat{O}} = \hat{A}_1^{\hat{A}_{n-1}^{\hat{O}}} = \hat{A}_1^{\hat{A}_1^{\cdot^{\cdot^{\cdot \hat{A}_1^{\hat{O}}}}}} \ . \tag{65}$$

Importantly, when we evaluate matrix elements by means of insertions of complete sets of states, this iterative structure is preserved (since all we do is to insert factors of unity).

Repeating the steps described in ref. [124], we can insert the above expression in the KMOC-expression and take the limit of localized massive states. The result is

$$\langle \Delta\hat{O} \rangle (p_1, p_2, b) = \int \frac{d^D q}{(2\pi)^{D-2}} \delta(2p_1 \cdot q - q^2) \delta(2p_2 \cdot q + q^2) e^{i\frac{b\cdot q}{\hbar}} \langle p_1' p_2' | \Delta O | p_1 p_2 \rangle \tag{66}$$

where $p_1' = p_1 - q$ and $p_2' = p_2 + q$. In this form it is clear that a first step is the evaluation of the matrix element $\langle p_1' p_2' | \Delta O | p_1 p_2 \rangle$, followed by the shown Fourier transform to b-space.

One noticeable feature of the KMOC-formalism for (non-spinning) black-hole scattering is that it always entails the evaluation of matrix elements of an operator (61) between two-particle scalar states. For an observable corresponding to an Hermitian operator $\hat{O}$ the corresponding ΔO is clearly Hermitian as well. Two-particle scalar matrix elements of this ΔO are then real, as follows from time-reversal symmetry. The reality of the expectation value is preserved by the insertion of the completeness relation since it just amounts to the insertion of factors of unity.

6 Quantum gravity corrections

So far we have discussed the classical gravitational radiation induced by the classical fluctuation of the space-time metric. When quantum field is considered these waves become quantized, carrying energy $\hbar\omega$ and spin $\pm 2\hbar$.

We have used that at low-energy a massive object, of mass M, behaves as a classical source interacting gravitationally through the fluctuations of the metric of the spacetime fabric. At large distance the spacetime differs from flat space by post-Newtonian corrections organized into a power series in the Schwarzschild radius $2G_N M/c^2$ over the distance r

$$g_{00} = 1 - 2\frac{G_N M}{c^2 r} + 2\frac{G_N^2 M^2}{c^4 r^2} + \cdots \tag{67}$$

which have been discussed in the Section 4 on the black hole metric.

The $\hbar$ expansion of the quantum scattering amplitudes in (9) contains positive powers of $\hbar$ that are quantum corrections. Some of these corrections are induced from the low energy part of the theory and are long range non-local effects, that do not depend on the ultraviolet completion of the theory.

Following the analysis of Section 4, but this time keeping some long-range quantum corrections to this metric (in harmonic gauge) we obtain [139, 160–162]

$$\delta g_{00} = \hbar\, \frac{62}{15\pi}\, \frac{G_N^2 M}{c^5 r^3} + \cdots \tag{68}$$

These quantum effects are not sensitive to high-energy gravitational effects because they arise solely from the quantum uncertainty on the gravitational field surrounding the massive source.

The logic described above is that of effective field theory. This technique provides a method for separating the known low-energy physics from the high-energy physics which may be either unknown or dynamically irrelevant. Indeed, in all of our fundamental theories we expect changes in the theory at the highest energy. Effective field theory method allows us to make reliable predictions using only the low-energy degrees of freedom and works even for theories that are deemed "non-renormalizable". In effective field theories we learn to focus on the predictions that follow from the reliable low energy end and such predictions often are non-local because of the Uncertainty Principle.

We illustrate how the techniques presented previous can lead to a derivation of the bending of light from a geodesic motion in the space curved by the Sun, and infrared quantum corrections.

The technique presented in the previous section applied to the scattering amplitude between a massive scalar of mass $M_\odot$ and a massless particle of spin S of energy ω has the low-energy limit, for small momentum transfer q^2, given by [127, 163]

$$\begin{aligned} i\mathcal{M}_S^{\text{tree}+1-\text{loop}} \simeq \frac{1}{\hbar}\,\frac{(M_\odot\omega)^2}{4}\Big[\frac{\kappa^2}{\vec q^{\,2}} + \kappa^4\frac{15}{512}\frac{M_\odot}{\sqrt{-\vec q^{\,2}}} \\ + \hbar\kappa^4\frac{15}{512\pi^2}\,\log\left(\frac{-\vec q^{\,2}}{M_\odot^2}\right) - \hbar\kappa^4\,\frac{bu(S)}{64\pi^2}\,\log\left(\frac{-\vec q^{\,2}}{\mu^2}\right) + \hbar\kappa^4\frac{3}{128\pi^2}\,\log^2\left(\frac{-\vec q^{\,2}}{\mu^2}\right) \\ - \kappa^4\,\frac{M_\odot\omega}{8\pi}\,\frac{i}{\vec q^{\,2}}\log\left(\frac{-\vec q^{\,2}}{M_\odot^2}\right)\Big]. \end{aligned} \tag{69}$$

where $bu(S)$ a constant depending on the spin S of the massless particle: $bu(0) = 3/40$ and $bu(1) = -161/120$.

The $\ln \vec{q}^{\,2}$ term arises from the loop calculation. It corresponds to the non-local effect and, if translated into a naïve bending angle using the formalism of [127] would result in

$$\chi \approx 4\frac{G_N M_\odot}{b} + \frac{15\pi}{4}\left(\frac{G_N M_\odot}{b}\right)^2 + + \frac{8bu(S) - 48\log(b/b_0)}{\pi}\frac{\hbar G_N^2 M_\odot}{b^3} \,. \tag{70}$$

The first two terms in χ reproduces the leading gravitational potential as well as its first post Newtonian correction to the gravitational potential in (67) computed by Einstein a century ago. They are organized in increasing powers of $G_N M_\odot/b$. They arise from the first line of (69), the last line is imaginary and does not contribute to the observables.

The last term in χ arises from the second line gives the quantum corrections to the gravitational interaction between the massless field and the massive source Part of these corrections arise from the uncertainty in the fluctuations of the virtual massless particles and gravitons. These corrections involve the product of the Planck length $\hbar$ and classical Schwarzschild radius $G_N M_\odot$ of the massive object. But there exists as well a dependence on the spin of the massless particle through the coefficient $bu(S)$ due to the delocalized nature of a massless field, which induces tidal like effects.

The possibility of embedding Einstein theory of gravity into an effective field theory framework is important because this open a systematic way of investigating the causal structure of scattering amplitudes in the eikonal regime including variety of contributions from quantum fluctuations from the dynamics of particles arising from either the standard model or from high energy completion [164–166]

For instance the quantum electrodynamics (QED) corrections to the scattering angle of a charged particle is [167]

$$\delta\chi = \frac{8G_N M_\odot}{b}\frac{\beta}{e}\left(\ln mb + \gamma_E - \ln 2\right) - \frac{4\beta G_N M_\odot}{E^2 b^2}, \tag{71}$$

which involves the β function for the particle running in the loop. Studying these effects, assuming analyticity and unitarity of the scattering amplitudes, give positivity constraints on the effective field theory which represent the imprint of causality on infrared observables.

7 Conclusion

Einstein theories of gravity has received a lot of experimental and observational confirmations [4]. But there are many reasons to think that this is not the ultimate theory of gravity. In this text we have presented an embedding of Einstein theory of gravity into an effective field theory framework.

It is therefore important to validate our current understanding of the connection between the quantum scattering amplitudes and classical general relativity in general dimensions [156,157]. By reproducing the classical Schwarzschild-Tangherlini metric from scattering amplitudes in four, five and six dimensions, we validate the procedure for extracting the classical piece from the quantum scattering amplitudes. The method can be applied to derive other black-hole metrics, like the Kerr-Newman and

Reissner-Nordström metrics by considering the vertex function of the emission of the graviton from a massive particle with spin and charge [98,100,101,119,139,168–170].

The scattering amplitude approach presented in this work can be applied to any effective field theory of gravity coupled to matter fields. One can include quantum corrections and examine the impact of quantum effects on the black-hole solutions [139], the effects of modified gravity models [171] or study the impact of higher derivative contributions [131–133] to the gravitational-wave templates.

The amplitudes computations, being performed in general dimensions, lead to results that have an analytic dependence on the space-time dimensions. It is remarkable that in this approach classical gravity physics contributions are determined by unitarity of the quantum amplitudes [74].

The advantage of this adaption of gravity is that it allows a direct unification with other fundamental forces at low energies in the context of the standard model. For instance, we verify the classical equivalence principle at the microscopic level by considering the scattering of different types of matter in the context of the bending of light around a huge massive star and demonstrate that the classical scattering angle is universal, as expected. But assuming unitarity and analyticity of the scattering amplitude lead to constraint on possible corrections to Einstein gravity from causality [164–166].

The raise of the quantum field theory approach to classical gravity has led to an improved understanding of the relation between general relativity and the quantum theory of gravity. This leads to many new exciting developments leading to a better understanding of the gravitational interactions in a binary system. This provides new techniques that can be applied to any gravitational effective field theories which have amplitude description: opening the possibility to search for deviation from Einstein gravity.

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