Quantum Gravity, Séminaire Poincaré XXIV (2023) 91 – 113 **Séminaire Poincaré**

The Double-Copy Approach to Gravity

Zvi Bern
Mani L. Bhaumik Institute for Theoretical Physics
Department of Physics and Astronomy, UCLA, Los Angeles, CA 90095, USA

Résumé.

Abstract : We summarize the double copy approach to perturbative gravity. This formulation is based on the idea that gravity is mediated by massless spin 2 gravitons, and that the complete perturbative interactions can be directly expressed in terms of products of spin 1 interactions. While equivalent to the standard geometric formulation of gravity, it is well suited for problems that can be phrased as perturbative expansions around flat space. We present two examples where the method has proven useful : high-order studies of the ultraviolet properties of supergravity theories and high-order calculations relevant for gravitational-wave physics.

1 Introduction

General relativity, as originally formulated by Albert Einstein, is a geometric theory formulated in the language of Riemannian geometry. As understood long ago by Richard Feynman [1], we can instead formulate gravity as a field theory of massless spin 2 gravitons. From this vantage point the principle of equivalence and the associated geometry become emergent phenomena in the classical limit. The idea of formulating gravity in the language closer of field theory was further advocated by Steven Weinberg, in the preface to his celebrated book on gravitation and cosmology [2] to emphasize the deep connection between gravitation and the other forces. This sentiment will echo throughout this lecture, showing that not only should gravity be thought of as a spiritual cousin of gauge theory, but that in detail the dynamics of gravity follows directly from gauge theory [3–6]. Gauge theories describe three of the four known forces, with gravity being the fourth one ; it is quite remarkable that all four forces can have a unified description. While this "double copy" formulation is inherently perturbative because it begins with the notion of spin 2 gravitons in flat space, it offers powerful insight to problems that fit into this framework. These ideas are most useful in the context of the S-matrix, which is defined in terms of incoming and outgoing particles in asymptotically flat space. The associated scattering amplitudes are independent of gauge or coordinate choices, making it much simpler to identify novel useful structures.

We will focus on two concrete examples : The first is the ultraviolet properties of supergravity theories and binary dynamics to two compact astrophysical objects in the context of gravitational-wave physics. There are a number of developments feed into this : Primary among them is the development of on-shell methods for computing scattering amplitudes. These methods are designed around the notion that we should focus on gauge-invariant quantities that have physical meaning. On-shell recursion [7] allows us to systematically build all tree-level scattering amplitudes in Einstein's theory or its supersymmetric extensions, starting only from the gauge-invariant on-shell three-point interactions. The generalized unitarity method [8–13] gives a systematic means for obtaining higher-order (loop) amplitudes from simpler tree amplitudes. A central idea is that gauge and gravity theories can be treated identically, and more surprisingly that one can obtain gravity scattering amplitudes directly from gauge-theory ones, without ever passing through the gravitational Lagrangian or equations of motion. The double copy relations between the gravity and gauge theory were first observed in string theory expressing closed-string (gravity) tree amplitudes to those of open strings (gauge theory) [3]. The relation between gravity and gauge theory becomes much simpler when re-phrased in terms of the so called duality between color and kinematics, also known as BCJ duality [4, 5].

The existence of such relations gives us new ways to think about gravity as well as to carry out nontrivial high-order calculations. One example is high-order studies of the ultraviolet properties of (super)gravity theories. Standard power counting following from the dimensionful nature of Newton's constant suggests that all point-like theories of gravity should to be ultraviolet divergent at some sufficiently high perturbative order. This is the nonrenomalizability of point-like theories of gravity. Of course, such arguments necessarily assume that all possible hidden symmetries have been taken into account properly. Otherwise, with unaccounted symmetries the derived bounds will be too weak, suggesting divergences when none are present. Conversely, if one were to discover the absence of divergences when there are no known symmetry reasons, it would strongly suggest the existence of a novel symmetry or structure. We can therefore view the problem of determining ultraviolet divergences in gravity theories as a search for new symmetries that would otherwise remain hidden. At present the only known means for carrying out the required multi-loop supergravity calculations are based on the double copy and the generalized unitarity method. This have been used for studies through five loops [14–19]. A key result of these calculations are the existence of *enhanced ultraviolet cancellations* for which are cancellations beyond the standard symmetry explanation. The case of $\mathcal{N} = 5$ supergravity is especially interesting : it does not diverge at four points at the four-loop order [18] despite there being no known symmetry mechanism protecting it [20, 21]. In this lecture we summarize the situation and point out an important calculation that should be carried out at the fifth loop order.

A second example where these ideas have proven fruitful is for gravitational-wave physics. This type of physics has risen to prominence because of the remarkable detection of gravitational waves [22], with a promise to fundamentally transform key areas in astronomy, cosmology, and particle physics. Moreover the anticipated increase in sensitivity of up to two orders in magnitude as well as sensitivity at much lower frequencies [23] offers a nontrivial challenge to theorists to produce predictions that match the unprecedented precision of the detectors. To properly meet the challenge multiple techniques will be needed to cover the full parameter space. This challenge has galvanized new work in multiple directions, including a program [24–26] for understanding the nature of gravitational-wave sources using ideas from quantum scattering amplitudes and effective field theory (EFT) [27]. The connection of scattering amplitudes to general relativity corrections to two-body interactions has long been understood [28, 29] and emphasized especially by Damour in Ref. [30]. This new effort utilizes basic tools from scattering amplitudes including generalized unitarity [8, 9, 11, 13], double-copy relations between gauge and gravity theories [3–6], and advanced multiloop integration [31–34]. These ideas have been used in various calculations advancing the state of the art through $\mathcal{O}(G^4)$ (with G being Newton's constant) for spinless gravitationally interacting binaries [25, 26, 35, 36]. (See also Refs. [37–42].) Recent progress suggests that the next order should be doable as well [43]. The scattering-amplitudes based approach complements traditional approaches to binary dynamics, such as effective one-body [44], numerical relativity [45], gravitational self-force [46], and the post-Newtonian (PN) approach (see e.g. Refs [27,37,47–49]). Since it naturally generates expansions at fixed order in G and all orders in velocity it naturally fits into the post-Minkowskian (PM) [30,50,51] approach. Here we will outline the methods and give a brief summary of some of the recent developments.

Although we will not discuss it here there have also been very interesting developments on understanding the double copy for nontrivial classical solutions and on understanding how various theories are linked via the double copy. We refer the reader to recent reviews [6, 52].

2 Double-copy relation between gravity and gauge theory.

Consider the Yang-Mills and Einstein-Hilbert Lagrangians,

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}\,, \qquad\qquad \mathcal{L}_{\mathrm{EH}} = \frac{2}{\kappa^2}\sqrt{-g}R\,. \tag{1}$$

These two Lagrangians have rather different properties. In particular with standard gauge choices, gauge theories have three- and four-point interactions, while gravity has an infinite number of contact interactions. Moreover, an inspection of the three-gluon and three-graviton interactions [53], reveal no simple connection between the two theories. Nevertheless, the double copy is a statement of precisely such simple relations in their scattering amplitudes.

The story of the double copy began in string theory. In the early day of string theory it was understood that at four points closed-string tree amplitudes could be expressed as a product of open-string amplitudes [54],

$$M(s,t,u) = \frac{\sin(\pi\alpha' s)}{\pi\alpha'} A(s,t)A(s,u)\,, \tag{2}$$

where α' is the inverse string tension and s, t, u are the usual Mandelstam variables, $s = (p_1 + p_2)^2$, $t = (p_1 + p_4)^2$, $u = (p_1 + p_3)^2$, where the p_i are the external momenta. Such relation hold for all string states, including the gluons of the open string and the gravitons of the closed string. At tree level, Kawai, Lewellen and Tye (KLT) [3] derived such relations for larger numbers of external particles. An immediate consequence is that in the low-energy limit where string theory reduced to field theory, one obtains relations between Einstein gravity and Yang-Mills tree-level scattering amplitudes for any number of external legs [10].

Subsequently, these relations were simplified using a duality between color and kinematics for gauge theories [4–6]. To describe this we first need to reorganize gauge-theory amplitudes. For simplicity here we discuss the case where all particles

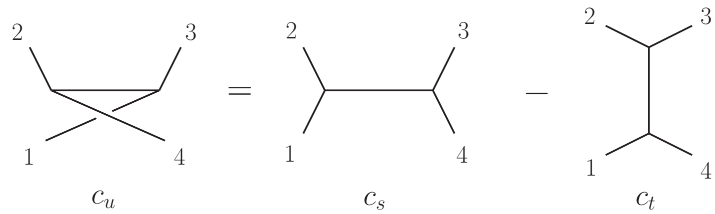


FIGURE 1 – The Jacobi relation at four points for the three channels labeled by s, t and u. These diagram represent either color factors or kinematic numerators.

are in the adjoint representation ; the case of fundamental representation particles works as well [55].

In general, we can write any m-point tree-level gauge-theory amplitude with all particles in the adjoint representation as,

$$\mathcal{A}^{\rm tree}(1,2,3,\ldots,m) = g^{m-2} \sum_i \frac{n_i\, c_i}{\prod_{\alpha_i} p^2_{\alpha_i}}\,, \tag{3}$$

where the sum runs over the set of m-point L-loop diagrams with only cubic vertices. These include distinct permutations of external legs. The product in the denominator runs over all propagators of each cubic diagram. The c_i are the color factors obtained by dressing every three vertex with an $\tilde{f}^{abc} = i\sqrt{2} f^{abc}$ structure constant, the n_i are kinematic numerator factors depending on momenta, polarizations and spinors and g is the coupling constant. The form (3) can be obtained straightforwardly, for example, from Feynman diagrams, by representing all contact terms as inverse propagators in the kinematic numerators that cancel propagators.

The c_i color factors satisfy Jacobi identities. In Fig. 1 the basic four-point identity is shown, where the diagrams represent color factors obtained by dressing each vertex with an $\tilde{f}^{abc}$. The duality requires there to exists a way of reorganizing the diagrams so that the numerators n_i satisfy equations in one-to-one correspondence with the Jacobi identities of the color factors. That is, for each triplet of color factors satisfying a Jacobi identity, the corresponding numerators satisfy a similar identity :

$$c_i = c_j - c_k \;\Rightarrow\; n_i = n_j - n_k\,. \tag{4}$$

The duality states that there exists representations of the amplitude, such that the color factors and numerators of the diagrams satisfy the relations. This duality holds to all multiplicity at tree level in a large variety of theories, including the important case of supersymmetric extensions of Yang-Mills theory. Fig. 1 displays the Jacobi relation at four points. The numerator relations are functional equations, depending nontrivially on the momenta and external polarizations and spinors. Beyond the four-point tree level, the relations are highly nontrivial and hold only after appropriate rearrangements of the amplitudes.

Explicit forms of tree-level amplitudes satisfying the duality have been given for an arbitrary number of external legs [56, 57]. An interesting consequence of the duality is that color-ordered partial tree amplitudes satisfy nontrivial relations [4, 58], proven in gauge theory and in string theory [59]. Although we do not yet have a satisfactory Lagrangian understanding, some progress in this direction can be found in Refs. [60–63]. A partial understanding of the underlying infinite-dimensional kine-

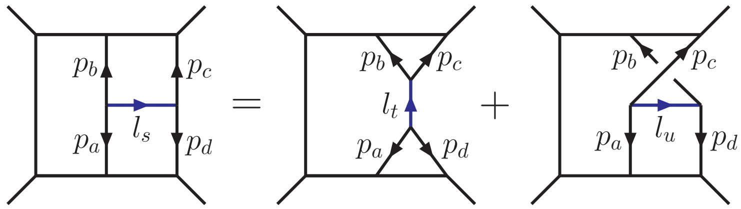


FIGURE 2 – An example of a duality relation satisfied by diagram numerators and color factors of the three-loop four-point amplitude. In this relation all but one propagator is identical in the three diagrams.

matics Lie algebra responsible for the duality is found in Refs. [61,63–66], although a complete understanding is still lacking.

Perhaps more surprising than the duality itself is a related conjecture that once gauge-theory amplitudes are put into a form satisfying the duality (4), corresponding gravity amplitudes are obtained simply by replacing the c_i color factor in Eq. (3) with a second copy of a numerator factor $\tilde{n}_i$ [4,5],

$$c_i \to \tilde{n}_i\,. \tag{5}$$

That is, the corresponding gravity amplitude i

$$\mathcal{M}^{\text{tree}}(1,2,\ldots,m) = i\Big(\frac{\kappa}{2}\Big)\sum_i \frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i} p^2_{\alpha_i}}\,, \tag{6}$$

where κ is the gravitational coupling. The n_j and $\tilde{n}_j$ are kinematic numerator factors from gauge-theory amplitudes, whose underlying theories can be different. The sum runs over the same set of diagrams with cubic vertices, as for gauge theory in Eq. (3). The double-copy formula hold, even if only one of the gauge-theory amplitudes has been put into a form that manifests the duality (4) [5,60].

The above relations are conjectured to hold at loop level [5] as well. Any m-point L-loop gauge-theory amplitude with all particles in the adjoint representation can be written in the form,

$$\mathcal{A}_m^{L-\text{loop}} = i^L g^{m-2+2L}\sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D}\frac{1}{S_j}\frac{n_j c_j}{\prod_{\alpha_j} p^2_{\alpha_j}}\,, \tag{7}$$

where the sum labeled by j runs over the set of distinct m-point L-loop graphs with only cubic vertices, including distinct relabelings of external legs. The factor S_j is the symmetry factor of graph j that removes overcounts from internal symmetry. In this representation contact terms are absorbed into graphs with cubic vertices by multiplying and dividing by appropriate inverse propagators. The integrals are over L independent D-dimensional loop momenta. The product in the denominator runs over all Feynman propagators of graph j. As at tree level, the c_j are the color factors of the diagrams, and the n_j are kinematic numerators of graph j depending on momenta, polarizations and spinors. According to the duality conjecture of Ref. [5], a representation of L-loop m-point amplitudes should exist where kinematic numerators satisfy the same algebraic properties as corresponding color factors

(4). As a simple example of a look level duality relation, Fig. 2 shows a three-loop relation that both numerators and color factors need to satisfy. The duality has been confirmed to hold in numerous cases at loop level in both supersymmetric and nonsupersymmetric (see e.g. Refs. [5, 5, 16, 57, 67–73]). Nevertheless, it remains a conjecture. While this complicates its applications at loop level, because the unitarity methods directly builds loop-level amplitudes from tree-level ones where the duality has been proven to hold, one can still use it as a powerful means for building loop integrands.

As at tree level, gravity amplitudes can be obtained from gauge-theory amplitudes simply by replacing the color factors with a second gauge-theory numerator factor, $c_j \to \tilde{n}_j$. That is, associated with the conjectured duality between color and kinematics is a double-copy formula for m-point L-loop gravity amplitudes [5]

$$\mathcal{M}_m^{\text{loop}} = i^{L+1}\left(\frac{\kappa}{2}\right)^{m-2+2L} \sum_j \int \prod_{l=1}^{L} \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_j} \frac{n_j \tilde{n}_j}{\prod_{\alpha_j} p_{\alpha_j}^2} \,. \tag{8}$$

The particular gravity theory obtained by the double-copy formula (8) is dictated by the choice of gauge theories. As at tree level, in Eq. (8) only one of the two copies needs to satisfy the duality (4) [5, 60]. The other gauge-theory amplitude can be in another convenient representation arranged into graphs with only cubic vertices.

At sufficiently high loop orders it becomes difficult to find forms of gauge theory amplitudes that manifest the BCJ numerator relations [74, 75]. For $\mathcal{N} = 4$ super-Yang-Mills four-point amplitudes this difficulty first occurs at five loops [74]. This can then greatly complicate the corresponding supergravity calculations because a simple double-copy procedure no longer holds. For a time the question of whether one could find BCJ representations of high-loop gauge-theory amplitudes became a major stumbling block to carrying out five-loop calculations in $\mathcal{N} = 8$ supergravity. This was solved by introducing the "generalized double copy" [76] where one can still apply the double copy, aided by a set of correction formulas that give missing contact contributions in the double copy whenever BCJ duality is not manifest. In this way, the five-loop four-point integrand of $\mathcal{N} = 8$ supergravity was successfully constructed [77] and its ultraviolet properties determined [19], as summarized in Sect. 4.

3 Generalized Unitarity Method

Unitarity has been a useful tool in quantum field theory since its inception. The Cutkosky rules [78] allow one to obtain the imaginary parts of one-loop amplitudes directly from products of tree amplitudes. (By imaginary we mean the absorptive part containing discontinuities across branch cuts.) This is generally substantially easier than a full diagrammatic calculation because one can greatly simplify the tree amplitudes before feeding them into cut calculations. Having obtained the imaginary parts, one traditionally uses dispersion relations to reconstruct real (dispersive) parts, up to additive rational function ambiguities. Here we describe an alternative integrand-based approach [8, 9, 11–13]. This approach avoids these issues by using on-shell quantities to directly construct an integrand equivalent the one obtained

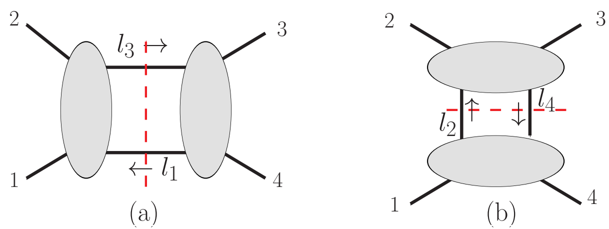


Figure 3 – The usual s- and t-channel two-particle cuts of the one-loop four-point amplitude, indicated by the dashed red lines. The exposed internal lines are all on-shell and integrated over phase space.

using Feynman diagrams. Because all quantities entering the calculation are on-shell it can be carried out in a gauge-invariant format.

To explain this approach, consider the s-channel cut of the four-point amplitude represented pictorially in Fig. 3a. The Mandelstam variables are as usual $s = (k_1 + k_2)^2$ and $t = (k_2 + k_3)^2$. According to the Cutkosky rules, the s-channel cut (with $s > 0$ and $t < 0$) of this amplitude is

$$-i\,\text{Disc}\,A_4(1,2,3,4)\Big|_{s\text{-cut}} = \int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\, 2\pi\delta^{(+)}(\ell_1^2)\, 2\pi\delta^{(+)}(\ell_3^2) \times A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_3)\, A_4^{\text{tree}}(-\ell_3, 3, 4, \ell_1)\,, \tag{9}$$

where $\ell_1 = p$ and $\ell_3 = p - k_1 - k_2$, $\delta^{(+)}$ is the positive-energy branch of the delta-function and 'Disc' means the discontinuity across the branch cut. Color-ordering requires us to maintain the clockwise ordering of the legs in sewing the tree amplitudes.

Suppose the amplitude had the form $A_4 = c\ln(-s) + \cdots = c(\ln|s| - i\pi) + \cdots$, where the coefficient c is a rational function. Then the phase space integral (9) would generate the $i\pi$ term but drop the $\ln|s|$ term. Since we wish to obtain both types of terms, real and imaginary, we replace the phase-space integral by the cut of an unrestricted loop momentum integral [8]; that is, we replace the δ-functions with Feynman propagators,

$$A_4(1,2,3,4)\Big|_{s\text{-cut}} = \left[\int \frac{d^{4-2\epsilon}p}{(2\pi)^{4-2\epsilon}}\, \frac{i}{\ell_1^2 + i\epsilon}\, A_4^{\text{tree}}(-\ell_1, 1, 2, \ell_3)\, \frac{i}{\ell_3^2 + i\epsilon}\, A_4^{\text{tree}}(-\ell_3, 3, 4, \ell_1)\right]\Bigg|_{s\text{-cut}}. \tag{10}$$

In contrast to Eq. (9) which includes only imaginary parts, Eq. (10) contains both real and imaginary parts. As indicated, Eq. (10) is valid only for those terms with an s-channel branch cut. Terms without an s-channel cut will in general not be correct, and need to be determined from other cuts. A key simplifying property of this formula is that one may continue to use on-shell conditions for the cut intermediate legs inside the tree amplitudes without affecting the result. Only terms containing no cut in this channel would change on application of the unitarity. A similar equation holds for the t-channel cut depicted in Fig. 3b. Combining the two cuts into a single function, one obtains the full amplitude, up to possible ambiguities in rational functions.

Cuts are usually taken as including phase-space integrals, but for our purposes it is simpler to define them as not including the phase-space integration. This procedure

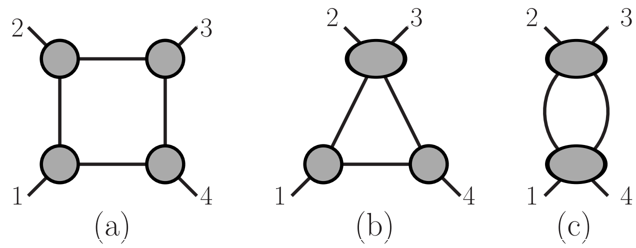


FIGURE 4 – The generalized cuts for a one-loop four-point amplitude organized according the method of maximal cuts. The cuts are organized according (a) maximal, (b) next-to-maximal and (c) next-to-next-to-maximal cuts. We dispense with the dashed lines to indicate cuts, and leave the integration for later. The complete set of cuts is obtained from the independent relabelings of external legs.

generalizes to an arbitrary number of external legs. To construct all terms with cuts in an amplitude, we combine the contributions from the various channels into a single function with the correct cuts in all channels. This procedure generalizes to all loop orders.

An especially useful class of generalized cuts are those that decompose a loop amplitude into a sum over m tree amplitudes of form,

$$C = \sum_{\text{states}} A^{\text{tree}}_{(1)} A^{\text{tree}}_{(2)} A^{\text{tree}}_{(3)} \cdots A^{\text{tree}}_{(m)} \,, \tag{11}$$

where the sum runs over all physical states that can cross the cuts. In $\mathcal{N}=4$ super-Yang-Mills theory, it is especially useful to consider the maximal cuts [12], where the maximum number of propagator lines are placed on shell. Other useful classes are single cuts where only a single internal line is placed on shell [79] or prescriptive unitarity [80, 81] which diagonalizes the method of maximal cuts.

In general, the complete amplitude is determined from a "spanning set" of cuts. Such sets are found by considering all potential independent contributions to the integrand that can enter an amplitude (and which do not integrate to zero), based on power counting or other constraints. One simply needs to ensure that all terms are non-vanishing in at least one cut that can then be used to determine its coefficient. In the $\mathcal{N}=4$ case one can often construct an ansatz for the entire amplitude using various conjectured properties. Once one has an ansatz, by confirming it over the spanning set, either numerically or analytically, we fully determine the ansatz.

One spanning set follows from the method of maximal cuts obtained [12] by starting from "maximal cuts", where the maximum numbers of internal propagators are placed on shell. The method of maximal cuts is quite helpful at higher-loop orders. In this method, the unitarity cuts (11) are arranged in levels according to the number k of internal propagators that remain off shell. As a simple example, the next-to-kth-maximaximal cuts N^{k}MC are illustrated for the one-loop four-point amplitude in Fig. 4 for $k = 0, 1, 2$ corresponding respectively to the cuts (a), (b) and (c). We refer to the generalized cut (a) as a maximal cut, cut (b) as a next-to-maximal cut and (c) as a next-to-next-to-maximal cut. This is an overcomplete set of cuts in the sense that cut (c) contains terms that are determined in cuts (a) and (b) and cut (b) contains terms found in cut (a). Each time a cut condition is released, potential contact terms which would not be visible at earlier steps are

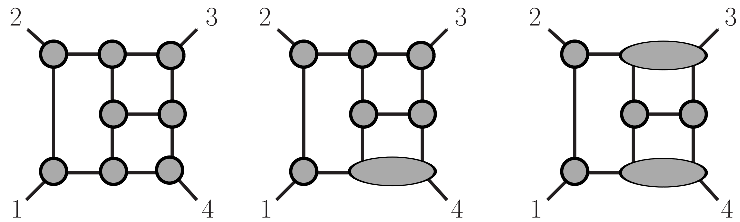


Figure 5 – Examples of generalized cuts for a three-loop four-point amplitude organized via the method of maximal cuts. The exposed lines are all on-shell and the blobs represent tree-level amplitudes.

captured. The process terminates when the only remaining potential contact terms exceed power counting requirements of the theory (or integrate to zero in dimensional regularization). It is useful to think of this as an upper triangular organization, where one first evaluate the maximal cuts, then subtracts out these terms from cut (b) and then subtract out the cut (a) and (b) terms from cut (c). This way of organizing the cuts has been used in a variety of applications in QCD [13] and supergravity, including for studies of ultraviolet properties of $\mathcal{N}=8$ supergravity through five loops [14, 16, 18] and in multiloop calculations of interest for gravitational wave. As a more complicated example, a variety of cuts organized via the method of maximal cuts are illustrated for a three-loop four-point amplitude in Fig. 5.

Given a spanning set of unitarity cuts, the task is to find an expression for the integrand of the amplitude with the correct cuts in all channels. This can be done either either via an ansatz whose arbitrary coefficients are determined by requiring that the expression has the correct cuts in all channels, or by systematically merging the cuts by adding missing pieces as each new cut is computed [82].

The method of maximal cuts can be applied just as well to either gauge or gravity amplitudes. As illustrated in the first diagram in Fig. 5, at the maximal cut (MC) level the maximum number of propagators are replaced by on-shell conditions and all tree amplitudes appearing in Eq. (11) are three-point amplitudes. At the next-to-maximal-cut (NMC) level, illustrated in the second cut of Fig. 5, a single propagator is placed off shell and so forth.

With this organization of generalized cuts, the integrands for L-loop amplitudes are obtained by first establishing an integrand whose maximal cuts are correct, then adding to it terms so that NMCs are all correct and systematically proceeding through the next^k maximal cuts (N^{k}MCs), until no further contributions are found. One straightforward way to implement this is by writing down an ansatz for diagrams of diagrams, which encompasses all possible terms that can appear in the integrand with an arbitrary parameter for each distinct term. Where this process completes is dictated by the power counting of the theory and by choices made at each level. For example, for four-point $\mathcal{N}=4$ sYM, maximal cuts are sufficient at one and two loops. For theories with less supersymmetry one needs deeper cuts. For example at one-loop in pure gravity one one needs up to N^{2}MC and at two loops one needs up to N^{4}MC.

Many calculations (see e.g. Refs. [14, 16–18, 68, 69, 75, 83]) find it convenient to organize the integrands in terms of diagrams with purely cubic vertices. Representations with only cubic diagrams have certain advantages : they are useful for establishing minimal power counting in each diagram, and the number of diagrams used to describe the result proliferate minimally with the loop order and multiplicity. A disadvantage is that ansätze are required for imposing various properties on each diagram, including the desired power counting, symmetry, and the multiple unitarity cuts to which a given diagram contributes. As the loop order increases, it becomes cumbersome to solve the requisite system of equations that imposes these constraints. One strategy for avoiding large systems is to use prescriptive unitarity [80,81]. For gravity calculations, the unitarity method is best used in conjunction with the double copy.

These methods are most effective when the particles are all massless or when wavefunction renormalization of the external particles is unimportant, such as in the classical limit. For cases, where the full mass dependence is required these methods need to be augmented with additional information [84–86] such as from known infrared and ultraviolet singularities, because of the appearance of terms such as $(m_i)^\epsilon$ which have no branch cuts in any channel.

For problems involving supersymmetric amplitudes, it can be very advantageous to use an on-shell superspace to evaluate unitarity cuts [87,88]. On-shell superspaces provides a convenient means for dealing with all states of the theory simultaneously, and for carrying out intermediate sums of states crossing cuts. On-shell superspaces organize the amplitudes according to physical helicity states. $\mathcal{N}=4$ super-Yang-Mills has a particularly simple structure because all states can be incorporated into a self CPT superfield,

$$\Phi(\eta) = g^+ + \eta^a f_a^+ + \frac{1}{2}\eta^a\eta^b\phi_{ab} + \frac{1}{3!}\epsilon_{abcd}\eta^a\eta^b\eta^c f^{d-} + \frac{1}{4!}\epsilon_{abcd}\eta^a\eta^b\eta^c\eta^d g^- \,. \tag{12}$$

Similar constructions hold also for theories with fewer supersymmetries [89] and for supergravity theories. The $\mathcal{N}=4$ superspace was applied by Nair [87] to the maximally helicity violating amplitudes. The $\mathcal{N}=4$ super-Yang-Mill maximally helicity violating amplitudes take the form

$$\mathcal{A}_n^{\rm MHV}(1,2,\cdots,n) = \frac{i}{\prod_{j=1}^n \langle j(j+1)\rangle}\,\delta^{(8)}\left(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a\right), \tag{13}$$

where the η_j^a are anti-commuting Grassmann parameters track the contributions from the different states, In Eq. (13) $n+1$ is to be identified with leg 1, and

$$\delta^{(8)}\left(\sum_{j=1}^n \lambda_j^\alpha \eta_j^a\right) = \prod_{a=1}^4 \sum_{i<j}^n \langle ij\rangle \eta_i^a \eta_j^a \,. \tag{14}$$

The component amplitudes are the coefficients in the η expansion of $\mathcal{A}_n$, with the external states identified according to their organization within the superfield, as in Eq. (12).

While here we do not go into details of super-amplitudes, for our purposes the key point is that it offers a simple way to track different particles crossing the unitarity cuts. In the context of unitarity he generalized $\mathcal{N}=4$ supercut is then given by simple Grassmann integrations which effectively performs the sum over states crossing the cuts,

$$\mathcal{C} = \int \Big[\prod_{i=1}^k d^4\eta_i\Big] \mathcal{A}_{(1)}^{\rm tree}\mathcal{A}_{(2)}^{\rm tree}\mathcal{A}_{(3)}^{\rm tree}\cdots\mathcal{A}_{(m)}^{\rm tree}\,, \tag{15}$$

where $\mathcal{A}^{\text{tree}}_{(j)}$ are the tree superamplitudes connected by k on-shell cut legs. Various efficient methods have been devised for evaluating the supersums that appear in unitarity cuts [90, 91]. One technical complication is that usually we are interested in computing loop amplitudes in the context of dimensional regularization, which is usually best dealt with by using higher-dimensional superspaces [92] and then dimensionally reducing to four dimensions [93].

4 Ultraviolet properties of supergravity theories

The study of the ultraviolet properties of theories of gravity has a long history, starting with the seminal work of 't Hooft and Veltman [94], who showed that pure Einstein gravity is finite at one loop but divergent in the presence of matter. Subsequently, pure Einstein gravity was shown to diverge at two loops [95, 96]. The ultraviolet behavior improves with the addition of supersymmetry. By the late 1970s it was clear that pure ungauged supergravities do not have divergences prior to three loops [97]. The consensus from studies in the 1980s was that all pure supergravity theories would likely diverge at the third loop order (see, for example, Ref. [98–100]). With additional assumptions on the existence of unconstrained superspaces one can raise the loop order of the predicted potential divergences [101] for $N \geq 5$ supergravities. It turns out such unconstrained superspaces do not exist. Nevertheless, as we shall see, cancellations extend even beyond these optimistic expectations [18].

The situation changed due to the ability to carry out high loop calculations directly showing finiteness of $\mathcal{N} = 8$ supergravity through four loops [14–16]. This led to more refined arguments based on supersymmetry and known duality symmetries reveal that ultraviolet divergences are delayed to surprisingly high-loop orders. In particular, such arguments show that $\mathcal{N} = 8$ supergravity is ultraviolet finite through at least six loops and $\mathcal{N} = 5$ supergravity through three loops [20, 102–105]. A natural question is then whether this is the final story or whether there is more to learn.

Happily, there are indeed more surprises. So far, the only explicitly calculated divergence for a pure supergravity is $\mathcal{N} = 4$ supergravity at four loops [17]. The interpretation of this divergence is, however, complicated by the presence of a $U(1)$ duality anomaly which might be behind its appearance [106, 107]. Such anomalies are absent in $\mathcal{N} \geq 5$ supergravities [108], making it is best to focus on these theories. Studies of unitarity cuts in $D = 4$ suggest the interesting possibility that divergences in $\mathcal{N} = 8$ supergravity may be further delayed [109, 110] in this dimension. The case of $\mathcal{N} = 5$ supergravity is especially interesting : it does not diverge at four points at the four-loop order [18] despite there being no known symmetry mechanism protecting it [20, 21].

Table 1 collects the consensus power counting results suggesting where the first first valid counterterms and possible divergences can occur in $\mathcal{N} = 4, 5, 8$ supergravity. These constraints includes using extended off-shell superpaces together with duality symmetries [20, 21, 102, 102, 104, 105], using the Berkovits pure spinor formalism [111] to fully expose supersymmetry [103], and using maximal cuts [12] of amplitudes to expose minimum powers of loop momenta that must appear in covariant Feynman-like diagrams [18, 112]. Each of these power counts give identical results. Of course, all assume that all symmetries have been identified and that there are no further hidden cancellations between diagrams. In the table, D is the space-time

Theory	Counterterm	Loop Order	divergence
$D = 4,\ Q = 32,\ \mathcal{N} = 8$	$\mathcal{D}^8 R^4$	7	unknown
$D = 4,\ Q = 16,\ \mathcal{N} = 4$	R^4	3	no
$D = 4,\ Q = 20,\ \mathcal{N} = 5$	$\mathcal{D}^2 R^4$	4	no
$D = 24/5,\ Q = 32$	$\mathcal{D}^8 R^4$	5	yes
$D = 5,\ Q = 16$	R^4	2	no

TABLE 1 – Counterterms corresponding to the first potential divergence that satisfy all proven supersymmetry and duality-symmetry constraints [20, 21, 102, 103, 105, 114]. The number of supercharges is Q and D is the space-time dimension. The fourth column shows the results of computing the coefficients of the divergences. In three cases the coefficients vanish and there is no divergence [18, 115, 116].

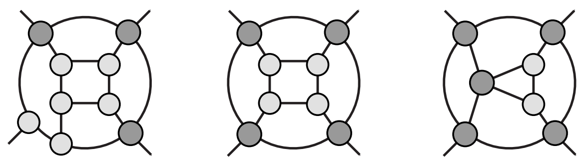


FIGURE 6 – Sample generalized cuts used in the construction of the five-loop four-point amplitude of $\mathcal{N} = 8$ supergravity. The exposed lines are all on-shell and the blobs represent tree-level amplitudes

dimensions and Q is the number of supercharges. Cancellations that occur beyond this cannot be manifested diagram by diagram whenever a covariant diagrammatic organization is used. Any further cancellations beyond the ones that occur within covariant diagrams are by definition *enhanced ultraviolet cancellations* [18]. For further details, see the recent review [113].

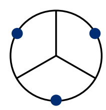
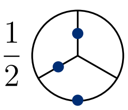
How do these power-counting symmetry constraints compared to the actual results obtained by direct calculations of the coefficients of potential divergences? As shown in the fourth column of Table 1 enhanced cancellation do indeed exist in multiple examples. This follows from a series of direct calculations of the divergences for $\mathcal{N} = 4, 5, 8$ [17–19, 115–117] supergravity that have shed considerable light on this question. These calculations are possible through use of the double copy and method of maximal cuts as described above in previous sections.

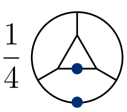
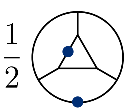
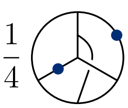
As a nontrivial example, at five loops a few of the cuts used in the calculation of the ultraviolet properties of $\mathcal{N} = 8$ are illustrated in Fig. 6. Table 2 summarizes the critical dimension where divergences actually first appear at a given loop order in $\mathcal{N} = 8$ supergravity, collecting the results of Refs. [15, 16, 19, 83, 98, 117]. The explicit values of the divergences that appear in the critical dimensions are are best summarized in terms of vacuum diagrams that encode the UV divergences. For example the divergences in the critical dimensions for three to five loops is explicitly

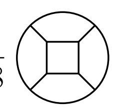
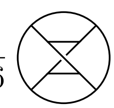
Loops	D_c for $\mathcal{N}=8$ sugra
1	8
2	7
3	6
4	11/2
5	24/5

TABLE 2 – The critical dimension D_c where ultraviolet divergences first occur in $\mathcal{N}=8$ supergravity, as determined by explicit calculations [20, 21, 102, 103, 105, 114].

given by

$\mathcal{M}_4^{(3)}\Big|_{\text{UV div.}} = -60\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^8 stu \left(\frac{1}{6}\right.$  $+ \frac{1}{2}$  $\left.\right),$

$\mathcal{M}_4^{(4)}\Big|_{\text{UV div.}} = -\frac{23}{2}\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^{10} (s^2+t^2+u^2)^2 \left(\frac{1}{4}\right.$  $+ \frac{1}{2}$  $+ \frac{1}{4}$  $\left.\right),$

$\mathcal{M}_4^{(5)}\Big|_{\text{UV div.}} = -\frac{16\times 629}{25}\,\mathcal{K}_{\text{G}} \left(\frac{\kappa}{2}\right)^{12} (s^2+t^2+u^2)^2 \left(\frac{1}{48}\right.$  $+ \frac{1}{16}$  $\left.\right),$ (16)

where the universal factor is $\mathcal{K}_{\text{G}} \equiv stu\, M_4^{\text{tree}}(1,2,3,4)$. and $s=(k_1+k_2)^2$, $t=(k_2+k_3)^2$ and $u=-s-t$ are the usual four-point Mandelstam invariants. The blue dots on the vacuum diagrams indicate that the corresponding propagator is squared. Through four loops analytic expressions exist for these integrals ; in any case, they directly demonstrate a nonvanishing divergence in the indicated critical dimension. To be well defined these vacuum integrals require a mass regulator ; the simplest choice is a uniform mass. It is of course quite remarkable that divergences have simple expressions in terms of vacuum diagrams. Note that the five-loop critical dimension is $D=24/5$, which requires an analytic continuation away from integer dimension, where the theory can be properly defined.

The most interesting outcome of these explicit calculations is the identification of enhanced cancellations in various cases. As the fourth column of Table 1 shows, in three of the listed cases cancellations appear that are *not* explained by the well studied power counting arguments, showing that enhanced cancellations do indeed exist. The case of $\mathcal{N}=4$ supergravity in $D=4$ does exhibit an enhanced cancellation at three loops as indicated in the table [115], but by four loops it does diverge [17]. Unfortunately, this theory suffers from a $U(1)$ anomaly, which then brings up the obvious question of whether the divergence is tied to this. The case of $\mathcal{N}=5$ supergravity at four loops is probably the most interesting example because it occurs in $D=4$ and it does not have an analogous $U(1)$ anomaly. For $\mathcal{N}=5$ supergravity, the apparent existence of a valid four-loop D^2R^4 counterterm [20, 21] suggests that a divergence should appear at four loops. As further explained in Ref. [20], these counterterms cannot be written as full-superspace integrals, but they do appear to respect all known standard-symmetries. However, as shown in Ref. [18] and summarized in Table 1 the potential divergence cancels nontrivially.

By increasing the space-time dimensions, one can lower the loop order at which

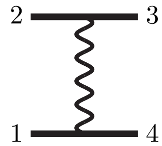


Figure 7 – The tree level scattering amplitude for two particles via a graviton. The straight lines represent massive scalars and wiggly lines are gravitons.

a divergence can first appear. In particular, half-maximal 16-supercharge supergravity in $D = 5$ exhibits a possible two-loop counterterm invariant under all known symmetries [114, 114]. This theory does not have an anomaly and is indeed finite at two loops despite the existence of an apparently valid counterterm. Very interestingly, the cancellation has been directly interpreted as a consequence of the double-copy structure. [116]. Unfortunately, this argument does not extend easily beyond one loop. For $\mathcal{N} = 8$ supergravity in $D = 4$, a counterterm is allowed at $L = 7$ loops [102, 102–105], putting it out of reach of currently available methods for carrying out multiloop calculations [118]. By taking an unphysical dimension of $D = 24/5$ for maximal 32-supercharge supergravity, corresponding to $\mathcal{N} = 8$ supergravity in $D = 4$, the loop order where a valid counterterm first appears is lowered from $L = 7$ to $L = 5$ [103]. In this case, the theory does diverge, as suggested by power counting, but one might wonder if working in an unphysical dimension might prevent nontrivial cancellations. Indeed, there is strong evidence from unitarity cuts that in $D = 4$ additional cancellations are present at seven loops and beyond [109, 110]. Various attempts and associated difficulties for putting tighter restrictions on the counterterms are found in Refs. [114, 119–124].

In light of the intriguing enhanced ultraviolet cancellations in Table 1 an obvious question is what calculations can be carried out that can help track down their origin. One possible tactic is to study the cancellations in unitarity cuts along the lines of Refs. [109,110]. An obvious direct calculation that would be very helpful would be to determine the coefficient of the potential $\mathcal{D}^4R^4$ $\mathcal{N} = 5$ supergravity counterterm, to see whether its coefficient vanishes. This is a particularly interesting calculation because if finite it would be the second enhanced cancellation for the theory. Very importantly, unlike for the potential seven-loop divergence of $\mathcal{N} = 8$ supergravity, this calculation is within reach of currently available methods. This calculation would, however, be more difficult than the calculation of $\mathcal{N} = 8$ supergravity at five loops.

5 Applications to gravitational waves

The discovery of gravitational waves [22] has stimulated a new direction in theoretical high-energy particle physics : using past advances in quantum scattering amplitudes to obtain new theoretical results for gravitational-wave physics. The scattering amplitudes-based approach derives classical binary dynamics taking advantage of Lorentz invariance, on-shell methods [7–9, 11, 13], double-copy relations between gauge and gravity theories [3–5], advanced multiloop integration [31–34], and EFT methods [27].

The post-Minkowskian approximation which maintains Lorentz invariance is the natural framework to apply scattering amplitude methods [30]. In the post-

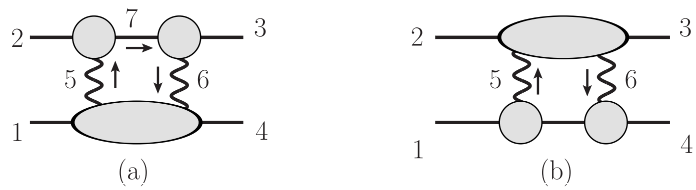


Figure 8 – The two generalized unitarity cuts for extracting the conservative two-body potential at 2PM order. The blobs represent tree amplitudes and exposed lines are all on shell.

Minkowskian framework one expands in Newton's constant but keeps all orders in velocity. An obvious first question is : what do quantum scattering amplitudes have to do with gravitational wave physics ? Perhaps the simplest way to see the link is via the observation that the leading order in Newton's constant G contribution to the two-body potential is precisely the Fourier transform of the appropriately normalized scattering amplitude. That is,

$$V(\boldsymbol{r},\boldsymbol{p}) = -\frac{8\pi G}{E_1 E_2}\int \frac{d^3 q}{2\pi^3}\,\mathcal{M}_4^{\text{tree}}(\boldsymbol{q},\boldsymbol{p})\,, \tag{17}$$

where the diagram giving the amplitude $\mathcal{M}_4^{\text{tree}}$ is depicted in Fig. 7. The tree level two-to-two gravitational scattering amplitude is simple enough to evaluate using either the Feynman diagram in Fig. 7 or using more advanced methods. In the center-of-mass this amplitude is given by

$$\mathcal{M}_4^{\text{tree}}(\boldsymbol{q},\boldsymbol{p}) = \frac{(p_1\cdot p_2)^2 - \frac{1}{2}m_1^2 m_2^2}{q^2} + O(q^0)\,, \tag{18}$$

where we drop any term that does not have a pole in q^2 because it corresponds to a short range contact interaction between the two objects ; here we are interested only in the long-range interactions. Performing the Fourier transform this results in a gravitational potential

$$V(\boldsymbol{r},\boldsymbol{p}) = -\frac{G}{|\boldsymbol{r}|}\frac{2(p_1\cdot p_2)^2 - m_1^2 m_2^2)}{E_1 E_2}\,. \tag{19}$$

In the static limit where $p_1 = (m_1, 0)$ and $p_2 = (m_2, 0)$ this reduces to Newton's potential,

$$V(\boldsymbol{r},0) = -\frac{G m_1 m_2}{|\boldsymbol{r}|}\,. \tag{20}$$

The PM expansion has received new attention in recent years (see e.g. [24–26,35, 36,125–131]). Beyond tree level the connection is more complicated, though the basic connection is essentially the same. As one goes to higher orders, the procedure for extracting the two-body Hamiltonian or physical observables becomes more complicated because of the need to either remove or cancel iteration pieces. These iteration pieces are uninteresting because they contain nothing but lower-order contributions which have presumably already been computed. There are various procedures for extracting observables. To extract a two-body Hamiltonian one uses and EFT matching, as explained in some detail in Refs. [24,26]. A more direct means is to extract the radial action [35], the eikonal phase [132],and other exponential representations

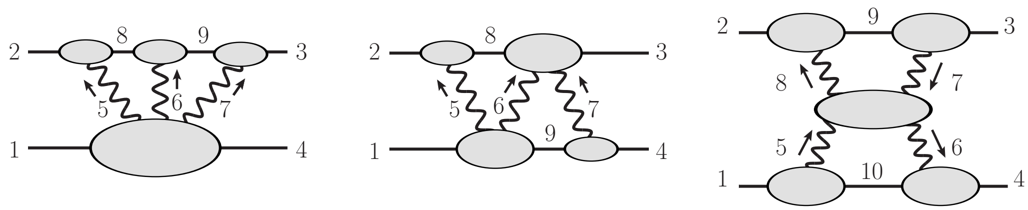


FIGURE 9 – The independent generalized cuts needed at $O(G^3)$ for the classical potential. The remaining contributing cuts are given by simple relabeling of external legs. Here the straight lines represent on-shell scalars and the wiggly lines correspond to on-shell gravitons or gluons.

of the amplitude [133], all of which can be used to obtain physical observables. One can also set up heavy mass expansions [134,135] to simplify the perturbative expansion. The Kosower, Maybee, O'Connell formalism directly gives observables [127].

The PM potential is given as an expansion in G,

$$V(\boldsymbol{p},\boldsymbol{r}) = \sum_{n=1}^{\infty} \left(\frac{G}{|\boldsymbol{r}|}\right)^n c_n(\boldsymbol{p}^2)\,, \tag{21}$$

where the coefficients c_n contain arbitrarily high powers in the velocity. The PM potential directly feeds into a two-body Hamiltonian, which in the center of mass is given by

$$H = \sqrt{\boldsymbol{p}^2 + m_1^2} + \sqrt{\boldsymbol{p}^2 + m_2^2} + V(\boldsymbol{p},\boldsymbol{r})\,, \tag{22}$$

where $\boldsymbol{p}$ and $-\boldsymbol{p}$ are the momenta of the two particles in the center of mass system. The masses are m_1 and m_2. Armed with the two-body Hamiltonian one can compute any observable. By expanding in velocity we recover the PN approximation. In the amplitudes based method one first starts with the generalized unitarity cuts. Figs. 8, 9 and 10 illustrate sample generalized cuts through $O(G^4)$. These are efficiently evaluated using the double copy.

The first calculations demonstrating the utility of these methods was the third order term in the conservative Hamiltonian(22) demonstrating that one can obtain useful results [25,26],

$$\begin{aligned}
c_1 &= \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right)\,,\\
c_2 &= \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1 - 2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1 - 2\sigma^2\right)^2}{2\gamma^3\xi^2}\right]\,,
\end{aligned}$$

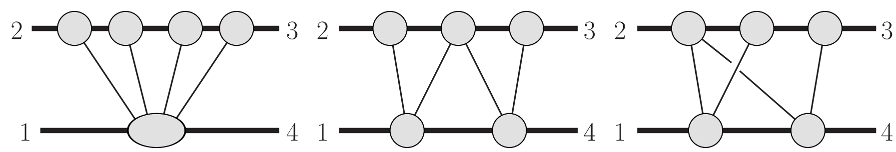


Figure 10 – Three of the thirteen generalized unitarity cuts for conservative contributions at $O(G^4)$. Exposed lines are on-shell. Thick lines represent massive scalars and thin lines are gravitons.

$$
\begin{aligned}
c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi} \Bigg[& \frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) \\
& - \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2 - 1}} - \frac{3\nu\gamma\left(1 - 2\sigma^2\right)\left(1 - 5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} \\
& - \frac{3\nu\sigma\left(7 - 20\sigma^2\right)}{2\gamma\xi} + \frac{2\nu^3(3 - 4\xi)\sigma\left(1 - 2\sigma^2\right)^2}{\gamma^4\xi^3} \\
& - \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1 - 2\sigma^2\right)}{4\gamma^3\xi^2} \\
& + \frac{\nu^4(1 - 2\xi)\left(1 - 2\sigma^2\right)^3}{2\gamma^6\xi^4} \Bigg] .
\end{aligned}
\tag{23}
$$

where we use center-of-mass coordinates where the incoming and outgoing particle momenta are $\pm\boldsymbol{p}$ and $\pm(\boldsymbol{p}-\boldsymbol{q})$, respectively. We define the total mass $m = m_1 + m_2$, the symmetric mass ratio $\nu = m_1 m_2/m^2$, the total energy $E = E_1 + E_2$, the symmetric energy ratio $\xi = E_1 E_2/E^2$, the energy-mass ratio $\gamma = E/m$, and the relativistic kinematic invariant $\sigma = p_1 \cdot p_2/m_1 m_2$. These results have been confirmed in various studies [37–41].

The scattering amplitudes approach to the two-body Hamiltonian has also been pushed to the fourth order in the coupling [35, 36]. By this order one encounters the tail effect [136] which introduces path dependence, with the net effect that the derived Hamiltonian is valid for large eccentricities but not small eccentricities [48]. The inability to smoothly use the large eccentricity Hamiltonian at small eccentricity is similar to the situation at the 4th post-Newtonian order. This is still to be resolved in the post-Minkowskian scattering setup which is always implicitly at large eccentricity.

Combining the results from up to the 4PM order [35, 36, 42, 130, 131] into an EOB-inspired resummation, gives impressive comparisons [137, 138] to the numerical relativity results of Ref. [139]. In particular, Fig. 11 shows the scattering angle as a function of angular momentum ; the agreement to numerical relativity is quite impressive, which strongly suggests that higher orders will continue to greatly improve the precision, to help meet the precision challenge of future gravitational wave observations. The methods outlined here make it straightforward to obtain the required integrands at the 5PM order and beyond. The nontrivial challenge, however, is to deal with the integration. Recent progress [43] based on carefully tuning the integration-by-parts [31] program FIRE [33] suggests that the 5PM order will be computed in the not too distant future.

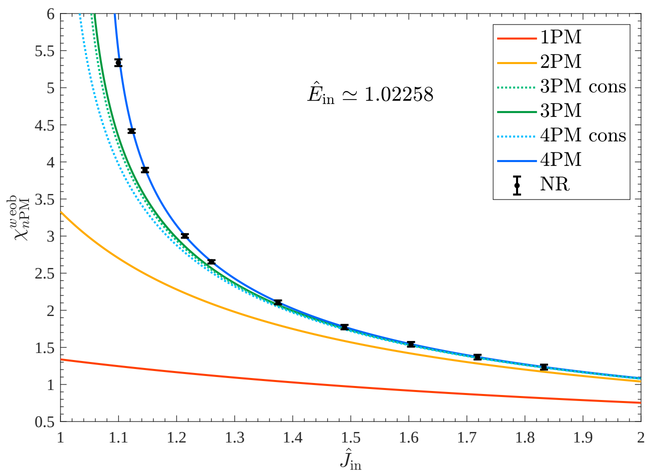


FIGURE 11 – Comparison of the EOB-inspired resummed scattering angle to numerical relativity as a function of angular momentum. The black points are numerical relativity results. From Ref. [137].

6 Outlook

In this lecture we presented an alternative approach to general relativity that starts from the notion that gravity is mediated by a massless spin 2 particle. It is a natural formalism for efficiently solving problems phrased as perturbative expansions in Newton's constant. It also fits well with the modern quantum amplitudes program based on unitarity allowing us to obtain higher orders by recycling lower orders. The double copy then give useful relations between gravity and gauge theory. This approach to gravity is well suited to high-order calculations. We presented two example where these methods are helpful. The first is higher-order calculations of ultraviolet properties of supergravity theories, demonstrating that there are *enhanced ultraviolet cancellations*, whose origin still needs to be explained. The second are high-order calculations relevant for gravitational-wave physics pushing forward the state of the art in the post-Minkowskian framework. This approach to perturbative general relativity is far from exhausted and in the coming years we can expect further calculations pushing forward the state of the art.

Références

[1] R. P. Feynman, F. B. Morinigo, W. G. Wagner and B. Hatfield, *Feynman lectures on gravitation/*

[2] S. Weinberg, *Gravitation and Cosmology : Principles and Applications of the General Theory of Relativity,* John Wiley and Sons, 1972, ISBN 978-0-471-92567-5, 978-0-471-92567-5

[3] H. Kawai, D. C. Lewellen and S. H. H. Tye, Nucl. Phys. B **269**, 1-23 (1986)

[4] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. D **78**, 085011 (2008) [arXiv :0805.3993 [hep-ph]].

[5] Z. Bern, J. J. M. Carrasco and H. Johansson, Phys. Rev. Lett. **105**, 061602 (2010) [arXiv :1004.0476 [hep-th]].

[6] Z. Bern, J. J. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, [arXiv :1909.01358 [hep-th]].

[7] R. Britto, F. Cachazo and B. Feng, Nucl. Phys. B **725**, 275-305 (2005) [arXiv :hep-th/0412103 [hep-th]].

[8] Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425**, 217-260 (1994) [arXiv :hep-ph/9403226 [hep-ph]]. Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **435**, 59-101 (1995) [arXiv :hep-ph/9409265 [hep-ph]].

[9] Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **513**, 3-86 (1998) [arXiv :hep-ph/9708239 [hep-ph]].

[10] Z. Bern, L. J. Dixon, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **546**, 423-479 (1999) [arXiv :hep-th/9811140 [hep-th]].

[11] R. Britto, F. Cachazo, B. Feng and E. Witten Phys. Rev. Lett. **94**, 181602 (2005) [arXiv :hep-th/0501052 [hep-th]].

[12] Z. Bern, J. J. M. Carrasco, H. Johansson and D. A. Kosower, Phys. Rev. D **76**, 125020 (2007) [arXiv :0705.1864 [hep-th]].

[13] C. F. Berger, Z. Bern, L. J. Dixon, F. Febres Cordero, D. Forde, H. Ita, D. A. Kosower and D. Maitre, Phys. Rev. D **78**, 036003 (2008) [arXiv :0803.4180 [hep-ph]].

[14] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson, D. A. Kosower and R. Roiban, Phys. Rev. Lett. **98**, 161303 (2007) [arXiv :hep-th/0702112 [hep-th]].

[15] Z. Bern, J. J. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. Lett. **103**, 081301 (2009) [arXiv :0905.2326 [hep-th]].

[16] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. D **85**, 105014 (2012) [arXiv :1201.5366 [hep-th]].

[17] Z. Bern, S. Davies, T. Dennen, A. V. Smirnov and V. A. Smirnov, Phys. Rev. Lett. **111**, no.23, 231302 (2013) [arXiv :1309.2498 [hep-th]].

[18] Z. Bern, S. Davies and T. Dennen, Phys. Rev. D **90**, no.10, 105011 (2014) [arXiv :1409.3089 [hep-th]].

[19] Z. Bern, J. J. Carrasco, W. M. Chen, A. Edison, H. Johansson, J. Parra-Martinez, R. Roiban and M. Zeng, Phys. Rev. D **98**, no.8, 086021 (2018) [arXiv :1804.09311 [hep-th]].

[20] G. Bossard, P. S. Howe, K. S. Stelle and P. Vanhove, Class. Quant. Grav. **28**, 215005 (2011) [arXiv :1105.6087 [hep-th]].

[21] D. Z. Freedman, R. Kallosh and Y. Yamada, Fortsch. Phys. **66**, no.10, 1800054 (2018) [arXiv :1807.06704 [hep-th]].

[22] B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. **116**, no. 6, 061102 (2016) [arXiv :1602.03837 [gr-qc]] ; B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], Phys. Rev. Lett. **119**, no. 16, 161101 (2017) [arXiv :1710.05832 [gr-qc]].

[23] M. Punturo *et al.,* "The Einstein Telescope : A third-generation gravitational wave observatory," Class. Quant. Grav. 27 (2010) 194002 ; P. Amaro-Seoane *et al.* [LISA], [arXiv :1702.00786 [astro-ph.IM]] ; D. Reitze *et al.* Bull. Am. Astron. Soc. **51**, 035 [arXiv :1907.04833 [astro-ph.IM]].

[24] C. Cheung, I. Z. Rothstein and M. P. Solon, Phys. Rev. Lett. **121**, no.25, 251101 (2018) [arXiv :1808.02489 [hep-th]].

[25] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon and M. Zeng, Phys. Rev. Lett. **122**, no.20, 201603 (2019) [arXiv :1901.04424 [hep-th]].

[26] Z. Bern, C. Cheung, R. Roiban, C. H. Shen, M. P. Solon and M. Zeng, JHEP **10**, 206 (2019) [arXiv :1908.01493 [hep-th]].

[27] W. D. Goldberger and I. Z. Rothstein, Phys. Rev. D **73**, 104029 (2006) [hep-th/0409156].

[28] Y. Iwasaki, Prog. Theor. Phys. **46**, 1587 (1971) ; Y. Iwasaki, Lett. Nuovo Cim. **1S2**, 783 (1971) [Lett. Nuovo Cim. **1**, 783 (1971)] ; S. N. Gupta and S. F. Radford, Phys. Rev. D **19**, 1065 (1979).

[29] J. F. Donoghue, Phys. Rev. D **50**, 3874-3888 (1994) [arXiv :gr-qc/9405057 [gr-qc]] ; N. E. J. Bjerrum-Bohr, J. F. Donoghue and B. R. Holstein, Phys. Rev. D **67**, 084033 (2003) [erratum : Phys. Rev. D **71**, 069903 (2005)] [arXiv :hep-th/0211072 [hep-th]] ; D. Neill and I. Z. Rothstein, Nucl. Phys. B **877**, 177-189 (2013) [arXiv :1304.7263 [hep-th]] ; N. E. J. Bjerrum-Bohr, J. F. Donoghue and P. Vanhove, JHEP **1402**, 111 (2014) o[arXiv :1309.0804 [hep-th]].

[30] T. Damour, Phys. Rev. D **97**, no.4, 044038 (2018) [arXiv :1710.10599 [gr-qc]].

[31] K.G. Chetyrkin and F.V. Tkachov, Nucl. Phys. B **192**, 159 (1981) ; S. Laporta, Int. J. Mod. Phys. A **15**, 5087-5159 (2000) [arXiv :hep-ph/0102033 [hep-ph]].

[32] A. V. Kotikov, Phys. Lett. B **254**, 158-164 (1991) ; Z. Bern, L. J. Dixon and D. A. Kosower, Nucl. Phys. B **412**, 751-816 (1994) [arXiv :hep-ph/9306240 [hep-ph]] ; E. Remiddi, Nuovo Cim. A **110**, 1435-1452 (1997) [arXiv :hep-th/9711188 [hep-th]] ; T. Gehrmann and E. Remiddi, Nucl. Phys. B **580**, 485-518 (2000) [arXiv :hep-ph/9912329 [hep-ph]].

[33] A. V. Smirnov, JHEP **10**, 107 (2008) [arXiv :0807.3243 [hep-ph]] ; A. V. Smirnov and F. S. Chuharev, [arXiv :1901.07808 [hep-ph]].

[34] J. Parra-Martinez, M. S. Ruf and M. Zeng, JHEP **11**, 023 (2020) [arXiv :2005.04236 [hep-th]].

[35] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C. H. Shen, M. P. Solon and M. Zeng, Phys. Rev. Lett. **126**, no.17, 171601 (2021) [arXiv :2101.07254 [hep-th]].

[36] Z. Bern, J. Parra-Martinez, R. Roiban, M. S. Ruf, C. H. Shen, M. P. Solon and M. Zeng, Phys. Rev. Lett. **128**, no.16, 161103 (2022) [arXiv :2112.10750 [hep-th]].

[37] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, Phys. Lett. B **807**, 135496 (2020) [arXiv :2003.07145 [gr-qc]].

[38] C. Cheung and M. P. Solon, JHEP **06**, 144 (2020) [arXiv :2003.08351 [hep-th]].

[39] D. Bini, T. Damour and A. Geralico, Phys. Rev. D **102**, no.2, 024062 (2020) [arXiv :2003.11891 [gr-qc]].

[40] G. Kälin, Z. Liu and R. A. Porto, [arXiv :2007.04977 [hep-th]].

[41] D. Bini, T. Damour and A. Geralico, Phys. Rev. D **102**, no.2, 024061 (2020) [arXiv :2004.05407 [gr-qc]].

[42] C. Dlapa, G. Kälin, Z. Liu and R. A. Porto, Phys. Rev. Lett. **128**, no.16, 161104 (2022) [arXiv :2112.11296 [hep-th]].

[43] Z. Bern, E. Herrmann, R. Roiban, M. S. Ruf, A. V. Smirnov, V. A. Smirnov and M. Zeng, [arXiv :2305.08981 [hep-th]].

[44] A. Buonanno and T. Damour, Phys. Rev. D **59**, 084006 (1999) [gr-qc/9811091] ; A. Buonanno and T. Damour, Phys. Rev. D **62**, 064015 (2000) [gr-qc/0001013].

[45] F. Pretorius, Phys. Rev. Lett. **95**, 121101 (2005) [gr-qc/0507014] ; M. Campanelli, C. O. Lousto, P. Marronetti and Y. Zlochower, Phys. Rev. Lett. **96**, 111101 (2006) [gr-qc/0511048] ; J. G. Baker, J. Centrella, D. I. Choi, M. Koppitz and J. van Meter, Phys. Rev. Lett. **96**, 111102 (2006) [gr-qc/0511103].

[46] Y. Mino, M. Sasaki and T. Tanaka, Phys. Rev. D **55**, 3457 (1997) [gr-qc/9606018] ; T. C. Quinn and R. M. Wald, Phys. Rev. D **56**, 3381 (1997) [gr-qc/9610053] ; E. Poisson, A. Pound and I. Vega, Living Rev. Rel. **14**, 7 (2011) [arXiv :1102.0529 [gr-qc]] ; L. Barack and A. Pound, Rept. Prog. Phys. **82**, no.1, 016904 (2019) [arXiv :1805.10385 [gr-qc]].

[47] J. Droste, Proc. Acad. Sci. Amst. 19 :44455 (1916) ; H. A. Lorentz and J. Droste, Koninklijke Akademie Van Wetenschappen te Amsterdam 26 392, 649 (1917). English translation in "Lorentz Collected papers," P. Zeeman and A. D. Fokker editors, Vol 5, 330 (1934-1939), The Hague : Nijhof ; A. Einstein, L. Infeld and B. Hoffmann, Annals Math. **39**, 65 (1938) ; T. Ohta, H. Okamura, T. Kimura and K. Hiida, Prog. Theor. Phys. **50**, 492 (1973) ; T. Damour and N. Deruelle, Phys. Lett. A **87**, 81 (1981) ; T. Damour, C.R. Acad. Sc. Paris, Série II, **294**, pp 1355-1357 (1982) ; P. Jaranowski and G. Schäfer, Phys. Rev. D **57**, 7274 (1998) Erratum : [Phys. Rev. D **63**, 029902 (2000)] [gr-qc/9712075] ; T. Damour, P. Jaranowski and G. Schäfer, Phys. Rev. D **62**, 044024 (2000) [gr-qc/9912092] ; L. Blanchet and G. Faye, Phys. Lett. A **271**, 58 (2000) [gr-qc/0004009] ; L. Blanchet and G. Faye, Phys. Rev. D **63**, 062005 (2001) [arXiv :gr-qc/0007051 [gr-qc]] ; V. C. de Andrade, L. Blanchet and G. Faye, Class. Quant. Grav. **18**, 753-778 (2001) [arXiv :gr-qc/0011063 [gr-qc]] ;

S. Foffa and R. Sturani, Phys. Rev. D **84**, 044031 (2011) [arXiv :1104.1122 [gr-qc]] ; S. Foffa and R. Sturani, Phys. Rev. D **87**, no.6, 064011 (2013) [arXiv :1206.7087 [gr-qc]] ;

T. Damour, P. Jaranowski and G. Schäfer, Phys. Rev. D **89**, no. 6, 064058 (2014) [arXiv :1401.4548 [gr-qc]] ; P. Jaranowski and G. Schäfer, Phys. Rev. D **92**, no. 12, 124043 (2015) [arXiv :1508.01016 [gr-qc]] ; L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat, Phys. Rev. D **93**, no.8, 084037 (2016) [arXiv :1512.02876 [gr-qc]] ; S. Foffa and R. Sturani, Phys. Rev. D **100**, no. 2, 024047 (2019) [arXiv :1903.05113 [gr-qc]] ; S. Foffa, R. A. Porto, I. Rothstein and R. Sturani, Phys. Rev. D **100**, no. 2, 024048 (2019) [arXiv :1903.05118 [gr-qc]] ;

S. Foffa, P. Mastrolia, R. Sturani and C. Sturm, Phys. Rev. D **95**, no. 10, 104009 (2017) [arXiv :1612.00482 [gr-qc]] ; L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat, Phys. Rev. D **93**, no.8, 084037 (2016) [arXiv :1512.02876 [gr-qc]] ; L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat, Phys. Rev. D **95**, no.4, 044026 (2017) [arXiv :1610.07934 [gr-qc]] ; L. Bernard, L. Blanchet, A. Bohé, G. Faye and S. Marsat, Phys. Rev. D **96**, no.10, 104043 (2017) [arXiv :1706.08480 [gr-qc]] ; T. Marchand, L. Bernard, L. Blanchet and G. Faye, Phys. Rev. D **97**, no.4, 044023 (2018) [arXiv :1707.09289 [gr-qc]] ; L. Bernard, L. Blanchet, G. Faye and T. Marchand, Phys. Rev. D **97**, no.4, 044037 (2018) [arXiv :1711.00283 [gr-qc]] ; J. Blümlein, A. Maier, P. Marquard and G. Schäfer, Nucl. Phys. B **955**, 115041 (2020) [arXiv :2003.01692 [gr-qc]] ; J. Blümlein, A. Maier, P. Marquard and G. Schäfer, Nucl. Phys. B **965**, 115352 (2021) [arXiv :2010.13672 [gr-qc]].

[48] D. Bini and T. Damour, Phys. Rev. D **96**, no.6, 064021 (2017) [arXiv :1706.06877 [gr-qc]].

[49] J. Blümlein, A. Maier, P. Marquard and G. Schäfer, Phys. Lett. B **816**, 136260 (2021) [arXiv :2101.08630 [gr-qc]].

[50] B. Bertotti, Nuovo Cimento **4** :898-906 (1956) R. P. Kerr, I. Nuovo Cimento **13** :469-491 (1959) ; B. Bertotti, J. F. Plebański, Ann, Phys, **11** :169-200 (1960) ; M. Portilla, J. Phys. A **12**, 1075 (1979) ; L. Bel, T. Damour, N. Deruelle, J. Ibanez and J. Martin, Gen. Rel. Grav. **13**, 963 (1981) ; K. Westpfahl, Fortschr. Phys., **33**, 417 (1985) ; T. Ledvinka, G. Schaefer and J. Bicak, Phys. Rev. Lett. **100**, 251101 (2008) [arXiv :0807.0214 [gr-qc]].

[51] T. Damour, Phys. Rev. D **94**, no. 10, 104015 (2016) [arXiv :1609.00354 [gr-qc]] ; N. E. J. Bjerrum-Bohr, P. H. Damgaard, G. Festuccia, L. Planté and P. Vanhove, Phys. Rev. Lett. **121**, no. 17, 171601 (2018) [arXiv :1806.04920 [hep-th]] ; A. Koemans Collado, P. Di Vecchia and R. Russo, Phys. Rev. D **100**, no.6, 066028 (2019) [arXiv :1904.02667 [hep-th]] ; A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. D **100**, no.8, 084040 (2019) [arXiv :1906.01579 [hep-th]] ; N. E. J. Bjerrum-Bohr, A. Cristofoli and P. H. Damgaard, JHEP **08**, 038 (2020) [arXiv :1910.09366 [hep-th]] ; P. H. Damgaard, K. Haddad and A. Helset, JHEP **11**, 070 (2019) [arXiv :1908.10308 [hep-ph]] ; A. Cristofoli, P. H. Damgaard, P. Di Vecchia and C. Heissenberg, JHEP **07**, 122 (2020) [arXiv :2003.10274 [hep-th]] ; G. Kälin and R. A. Porto, JHEP **11**, 106 (2020) [arXiv :2006.01184 [hep-th]] ; G. Mogull, J. Plefka and J. Steinhoff, JHEP **02**, 048 (2021) [arXiv :2010.02865 [hep-th]] ; D. Bonocore, A. Kulesza and J. Pirsch, [arXiv :2112.02009 [hep-th]].

[52] T. Adamo, J. J. M. Carrasco, M. Carrillo-González, M. Chiodaroli, H. Elvang, H. Johansson, D. O'Connell, R. Roiban and O. Schlotterer, [arXiv :2204.06547 [hep-th]].

[53] B. S. DeWitt, Phys. Rev. **162**, 1239-1256 (1967)

[54] M. B. Green, J. H. Schwarz and E. Witten, 1988, ISBN 978-0-521-35752-4

[55] H. Johansson and A. Ochirov, JHEP **11**, 046 (2015) [arXiv :1407.4772 [hep-th]].

[56] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, JHEP **1101**, 001 (2011) [1010.3933 [hep-th]] ; C. R. Mafra, O. Schlotterer and S. Stieberger, 1104.5224 [hep-th] ; A. Edison and F. Teng, JHEP **12**, 138 (2020) doi :10.1007/JHEP12(2020)138 [arXiv :2005.03638 [hep-th]].

[57] A. Edison, S. He, H. Johansson, O. Schlotterer, F. Teng and Y. Zhang, JHEP **02**, 164 (2023) [arXiv :2211.00638 [hep-th]] ; A. Edison, S. He, O. Schlotterer and F. Teng, JHEP **09**, 079 (2020) [arXiv :2005.03639 [hep-th]].

[58] N. E. J. Bjerrum-Bohr, P. H. Damgaard, T. Sondergaard and P. Vanhove, arXiv :1003.2403 [hep-th] ; S.-H. H. Tye and Y. Zhang, JHEP **1006**, 071 (2010) [arXiv :1003.1732 [hep-th]].

[59] N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. Lett. **103**, 161602 (2009) [0907.1425 [hep-th]] ; S. Stieberger, 0907.2211 [hep-th]. B. Feng, R. Huang and Y. Jia, Phys. Lett. B **695**, 350 (2011) [arXiv :1004.3417 [hep-th]]. Y. X. Chen, Y. J. Du and B. Feng, JHEP **1102**, 112 (2011) [1101.0009 [hep-th]] ; F. Cachazo, arXiv :1206.5970 [hep-th].

[60] Z. Bern, T. Dennen, Y. t. Huang and M. Kiermaier, Phys. Rev. D **82**, 065003 (2010) [arXiv :1004.0693 [hep-th]].

[61] R. Monteiro and D. O'Connell, JHEP **07**, 007 (2011) [arXiv :1105.2565 [hep-th]].

[62] C. Cheung, G. N. Remmen, C. H. Shen and C. Wen, JHEP **04**, 129 (2018) [arXiv :1709.04932 [hep-th]].

[63] C. Cheung, J. Mangan, J. Parra-Martinez and N. Shah, Phys. Rev. Lett. **129**, no.22, 22 (2022) [arXiv :2204.07130 [hep-th]].

[64] N. E. J. Bjerrum-Bohr, P. H. Damgaard, R. Monteiro and D. O'Connell, JHEP **06**, 061 (2012) [arXiv :1203.0944 [hep-th]].

[65] G. Chen, G. Lin and C. Wen, Phys. Rev. D **107**, no.8, L081701 (2023) [arXiv :2208.05519 [hep-th]].

[66] G. Chen, L. Rodina and C. Wen, [arXiv :2310.11943 [hep-th]].

[67] J. J. M. Carrasco and H. Johansson, Phys. Rev. D **85**, 025006 (2012) [arXiv :1106.4711 [hep-th]].

[68] R. H. Boels, B. A. Kniehl, O. V. Tarasov and G. Yang, JHEP **02**, 063 (2013) [arXiv :1211.7028 [hep-th]].

[69] G. Yang, Phys. Rev. Lett. **117**, no.27, 271602 (2016) [arXiv :1610.02394 [hep-th]].

[70] G. Lin, G. Yang and S. Zhang, JHEP **03**, 061 (2022) [arXiv :2111.03021 [hep-th]].

[71] J. J. M. Carrasco and I. A. Vazquez-Holm, JHEP **11**, 088 (2021) [arXiv :2108.06798 [hep-th]].

[72] Z. Li, G. Yang and J. Zhang, Commun. Theor. Phys. **74**, no.6, 065203 (2022) [arXiv :2204.09407 [hep-th]].

[73] J. J. M. Carrasco and A. Seifi, JHEP **05**, 217 (2023) [arXiv :2302.14861 [hep-th]].

[74] Z. Bern, J. J. M. Carrasco, H. Johansson and R. Roiban, Phys. Rev. Lett. **109**, 241602 (2012) [arXiv :1207.6666 [hep-th]].

[75] G. Mogull and D. O'Connell, JHEP **12**, 135 (2015) [arXiv :1511.06652 [hep-th]].

[76] Z. Bern, J. J. Carrasco, W. M. Chen, H. Johansson and R. Roiban, Phys. Rev. Lett. **118**, no.18, 181602 (2017) [arXiv :1701.02519 [hep-th]].

[77] Z. Bern, J. J. M. Carrasco, W. M. Chen, H. Johansson, R. Roiban and M. Zeng, Phys. Rev. D **96**, no.12, 126012 (2017) [arXiv :1708.06807 [hep-th]].

[78] R. E. Cutkosky, J. Math. Phys. **1**, 429-433 (1960)

[79] E. W. Nigel Glover and C. Williams, JHEP **0812**, 067 (2008) [0810.2964 [hep-th]]. I. Bierenbaum, S. Buchta, P. Draggiotis, I. Malamos and G. Rodrigo, JHEP **03**, 025 (2013) [arXiv :1211.5048 [hep-ph]] ; S. Caron-Huot, JHEP **05**, 080 (2011) [arXiv :1007.3224 [hep-ph]] ; N. Arkani-Hamed, J. L. Bourjaily, F. Cachazo, S. Caron-Huot and J. Trnka, JHEP **01**, 041 (2011) [arXiv :1008.2958 [hep-th]].

[80] J. L. Bourjaily, E. Herrmann and J. Trnka, JHEP **06**, 059 (2017) [arXiv :1704.05460 [hep-th]].

[81] J. L. Bourjaily, E. Herrmann, C. Langer, A. J. McLeod and J. Trnka, JHEP **12**, 073 (2019) [arXiv :1909.09131 [hep-th]].

[82] Z. Bern, L. J. Dixon and D. A. Kosower, JHEP **08**, 012 (2004) [arXiv :hep-ph/0404293 [hep-ph]].

[83] Z. Bern, J. J. M. Carrasco, L. J. Dixon, H. Johansson and R. Roiban, Phys. Rev. D **78**, 105019 (2008) [arXiv :0808.4112 [hep-th]].

[84] Z. Bern and A. G. Morgan, Nucl. Phys. B **467**, 479-509 (1996) [arXiv :hep-ph/9511336 [hep-ph]].

[85] R. Britto and E. Mirabella, JHEP **01**, 045 (2012) [arXiv :1109.5106 [hep-ph]].

[86] Z. Bern, D. Kosmopoulos and A. Zhiboedov, J. Phys. A **54**, no.34, 344002 (2021) [arXiv :2103.12728 [hep-th]].

[87] V. P. Nair, Phys. Lett. B **214**, 215 (1988).

[88] M. Bianchi, H. Elvang and D. Z. Freedman, JHEP **0809**, 063 (2008) [0805.0757 [hep-th]] ; M. Kiermaier, H. Elvang and D. Z. Freedman, JHEP **0906**, 068 (2009) [0811.3624 [hep-th]] ; M. Kiermaier and S. G. Naculich, JHEP **0905**, 072 (2009) [0903.0377 [hep-th]].

[89] H. Elvang, Y. t. Huang and C. Peng, JHEP **09**, 031 (2011) [arXiv :1102.4843 [hep-th]].

[90] H. Elvang, D. Z. Freedman and M. Kiermaier, JHEP **0904**, 009 (2009) [0808.1720 [hep-th]].

[91] Z. Bern, J. J. M. Carrasco, H. Ita, H. Johansson and R. Roiban, Phys. Rev. D **80**, 065029 (2009) [0903.5348 [hep-th]].

[92] T. Dennen, Y. t. Huang and W. Siegel, JHEP **1004**, 127 (2010) [0910.2688 [hep-th]]. A. Brandhuber, D. Korres, D. Koschade and G. Travaglini, JHEP **02**, 077 (2011) [arXiv :1010.1515 [hep-th]].

[93] Z. Bern, J. J. Carrasco, T. Dennen, Y. t. Huang and H. Ita, Phys. Rev. D **83**, 085022 (2011) [arXiv :1010.0494 [hep-th]].

[94] G. 't Hooft and M. J. G. Veltman, Ann. Inst. H. Poincare Phys. Theor. A **20**, 69-94 (1974)

[95] M. H. Goroff and A. Sagnotti, Nucl. Phys. B **266**, 709-736 (1986) ; A. E. M. van de Ven, Nucl. Phys. B **378**, 309-366 (1992)

[96] Z. Bern, C. Cheung, H. H. Chi, S. Davies, L. Dixon and J. Nohle, Phys. Rev. Lett. **115**, no.21, 211301 (2015) [arXiv :1507.06118 [hep-th]] ; Z. Bern, H. H. Chi, L. Dixon and A. Edison, Phys. Rev. D **95**, no.4, 046013 (2017) [arXiv :1701.02422 [hep-th]].

[97] M. T. Grisaru, Phys. Lett. B **66**, 75-76 (1977) E. Tomboulis, Phys. Lett. B **67**, 417-420 (1977) S. Deser and J. H. Kay, Phys. Lett. B **76**, 400-403 (1978)

[98] M. B. Green, J. H. Schwarz and L. Brink, Nucl. Phys. B **198**, 474-492 (1982)

[99] N. Marcus and A. Sagnotti, Nucl. Phys. B **256**, 77-108 (1985)

[100] P. S. Howe and K. S. Stelle, Int. J. Mod. Phys. A **4**, 1871 (1989)

[101] M. T. Grisaru and W. Siegel, Nucl. Phys. B **201**, 292 (1982) [erratum : Nucl. Phys. B **206**, 496 (1982)]

[102] M. B. Green, J. G. Russo and P. Vanhove, JHEP **06**, 075 (2010) [arXiv :1002.3805 [hep-th]].

[103] J. Bjornsson and M. B. Green, JHEP **08**, 132 (2010) [arXiv :1004.2692 [hep-th]].

[104] G. Bossard, P. S. Howe and K. S. Stelle, JHEP **01**, 020 (2011) [arXiv :1009.0743 [hep-th]].

[105] N. Beisert, H. Elvang, D. Z. Freedman, M. Kiermaier, A. Morales and S. Stieberger, Phys. Lett. B **694**, 265-271 (2011) [arXiv :1009.1643 [hep-th]].

[106] Z. Bern, J. Parra-Martinez and R. Roiban, Phys. Rev. Lett. **121**, no.10, 101604 (2018) [arXiv :1712.03928 [hep-th]].

[107] Z. Bern, D. Kosower and J. Parra-Martinez, Proc. Roy. Soc. Lond. A **476**, no.2235, 20190722 (2020) [arXiv :1905.05151 [hep-th]].

[108] D. Z. Freedman, R. Kallosh, D. Murli, A. Van Proeyen and Y. Yamada, JHEP **05**, 067 (2017) [arXiv :1703.03879 [hep-th]].

[109] E. Herrmann and J. Trnka, JHEP **02**, 084 (2019) [arXiv :1808.10446 [hep-th]].

[110] A. Edison, E. Herrmann, J. Parra-Martinez and J. Trnka, SciPost Phys. **10**, no.1, 016 (2021) [arXiv :1909.02003 [hep-th]].

[111] N. Berkovits, JHEP **04**, 018 (2000) [arXiv :hep-th/0001035 [hep-th]].

[112] Z. Bern, M. Enciso, J. Parra-Martinez and M. Zeng, JHEP **05**, 137 (2017) [arXiv :1703.08927 [hep-th]].

[113] Z. Bern, J. J. M. Carrasco, M. Chiodaroli, H. Johansson and R. Roiban, [arXiv :2304.07392 [hep-th]].

[114] G. Bossard, P. S. Howe and K. S. Stelle, JHEP **07**, 117 (2013) [arXiv :1304.7753 [hep-th]].

[115] Z. Bern, S. Davies, T. Dennen and Y. t. Huang, Phys. Rev. Lett. **108**, 201301 (2012) [arXiv :1202.3423 [hep-th]].

[116] Z. Bern, S. Davies, T. Dennen and Y. t. Huang, Phys. Rev. D **86**, 105014 (2012) [arXiv :1209.2472 [hep-th]].

[117] Z. Bern, L. J. Dixon, D. C. Dunbar, M. Perelstein and J. S. Rozowsky, Nucl. Phys. B **530**, 401-456 (1998) [arXiv :hep-th/9802162 [hep-th]].

[118] J. J. M. Carrasco, A. Edison and H. Johansson, [arXiv :2112.05178 [hep-th]].

[119] G. Bossard, P. S. Howe and K. S. Stelle, Phys. Lett. B **719**, 424-429 (2013) [arXiv :1212.0841 [hep-th]].

[120] Z. Bern, S. Davies and T. Dennen, Phys. Rev. D **88**, 065007 (2013) [arXiv :1305.4876 [hep-th]].

[121] R. Kallosh, JHEP **05**, 109 (2019) [arXiv :1812.08087 [hep-th]].

[122] M. Gunaydin and R. Kallosh, JHEP **09**, 105 (2019) [arXiv :1812.08758 [hep-th]].

[123] J. J. M. Carrasco, M. Lewandowski and N. H. Pavao, Phys. Rev. Lett. **131**, no.5, 051601 (2023) [arXiv :2203.03592 [hep-th]].

[124] R. Kallosh, [arXiv :2304.13211 [hep-th]] ; R. Kallosh, [arXiv :2304.13926 [hep-th]] ; R. Kallosh, [arXiv :2311.10084 [hep-th]].

[125] T. Damour, "Gravitational scattering, post-Minkowskian approximation and effective one-body theory," Phys. Rev. D **94**, no. 10, 104015 (2016) [arXiv :1609.00354 [gr-qc]] ;
J. Feng, M. Baumann, B. Hall, J. Doss, L. Spencer and R. Matzner, "PoMiN : A Post-Minkowskian N-Body Solver," Astrophys. J. **859**, no. 2, 130 (2018) doi :10.3847/1538-4357/aac209 [arXiv :1805.00813 [gr-qc]] ;
L. Blanchet and A. S. Fokas, "Equations of motion of self-gravitating N-body systems in the first post-Minkowskian approximation," Phys. Rev. D **98**, no. 8, 084005 (2018) doi :10.1103/PhysRevD.98.084005 [arXiv :1806.08347 [gr-qc]].

[126] D. Bini and T. Damour, "Gravitational spin-orbit coupling in binary systems, post-Minkowskian approximation and effective one-body theory," Phys. Rev. D **96**, no. 10, 104038 (2017) [arXiv :1709.00590 [gr-qc]] ;
J. Vines, "Scattering of two spinning black holes in post-Minkowskian gravity, to all orders in spin, and effective-one-body mappings," Class. Quant. Grav. **35**, no. 8, 084002 (2018) doi :10.1088/1361-6382/aaa3a8 [arXiv :1709.06016 [gr-qc]] ;
A. Guevara, A. Ochirov and J. Vines, "Scattering of spinning black holes from exponentiated soft factors," arXiv :1812.06895 [hep-th] ;
J. Vines, J. Steinhoff and A. Buonanno, "Spinning-black-hole scattering and the test-black-hole limit at second post-Minkowskian order," Phys. Rev. D **99**, no. 6, 064054 (2019) doi :10.1103/PhysRevD.99.064054 [arXiv :1812.00956 [gr-qc]] ;
M. Z. Chung, Y. T. Huang, J. W. Kim and S. Lee, "The simplest massive S-matrix : from minimal coupling to Black Holes," JHEP **1904**, 156 (2019) [arXiv :1812.08752 [hep-th]] ;
B. Maybee, D. O'Connell, J. Vines, "Observables and amplitudes for spinning particles and black holes," arXiv :1906.09260 [hep-th] ;
A. Guevara, A. Ochirov and J. Vines, "Black-hole scattering with general spin directions from minimal-coupling amplitudes," arXiv :1906.10071 [hep-th].

[127] D. A. Kosower, B. Maybee and D. O'Connell, JHEP **02**, 137 (2019) [arXiv :1811.10950 [hep-th]].

[128] A. Koemans Collado, P. Di Vecchia and R. Russo, Phys. Rev. D **100**, no.6, 066028 (2019) [arXiv :1904.02667 [hep-th]].

[129] A. Cristofoli, N. E. J. Bjerrum-Bohr, P. H. Damgaard and P. Vanhove, Phys. Rev. D **100**, no.8, 084040 (2019) [arXiv :1906.01579 [hep-th]].

[130] A. V. Manohar, A. K. Ridgway and C. H. Shen, Phys. Rev. Lett. **129**, no.12, 121601 (2022) [arXiv :2203.04283 [hep-th]].

[131] C. Dlapa, G. Kälin, Z. Liu, J. Neef and R. A. Porto, Phys. Rev. Lett. **130**, no.10, 101401 (2023) [arXiv :2210.05541 [hep-th]].

[132] R. J. Glauber, in "Lectures in theoretical physics", ed. by W. E. Brittin and L. G. Dunham, Interscience Publishers, Inc., New York, Volume I, page 315, (1959) ; D. Amati, M. Ciafaloni and G. Veneziano, Nucl. Phys. B **347**, 550 (1990) ; R. Saotome and R. Akhoury, JHEP **01**, 123 (2013) [arXiv :1210.8111 [hep-th]] ; R. Akhoury, R. Saotome and G. Sterman, arXiv :1308.5204 [hep-th] ; P. Di Vecchia, A. Luna, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, Phys. Lett. B **798**, 134927 (2019) [arXiv :1908.05603 [hep-th]] ; P. Di Vecchia, S. G. Naculich, R. Russo, G. Veneziano and C. D. White, JHEP **03**, 173 (2020) [arXiv :1911.11716 [hep-th]] ; Z. Bern, H. Ita, J. Parra-Martinez and M. S. Ruf, Phys. Rev. Lett. **125**, no.3, 031601 (2020) [arXiv :2002.02459 [hep-th]] ; P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, Phys. Lett. B **811**, 135924 (2020) [arXiv :2008.12743 [hep-th]] ; P. Di Vecchia, C. Heissenberg, R. Russo and G. Veneziano, [arXiv :2101.05772 [hep-th]].

[133] P. H. Damgaard, L. Plante and P. Vanhove, JHEP **11**, 213 (2021) [arXiv :2107.12891 [hep-th]].

[134] R. Aoude, K. Haddad and A. Helset, JHEP **05**, 051 (2020) [arXiv :2001.09164 [hep-th]].

[135] A. Brandhuber, G. Chen, G. Travaglini and C. Wen, JHEP **07**, 047 (2021) [arXiv :2104.11206 [hep-th]].

[136] W. Bonnor, Philos. Trans. R. Soc. London, Ser. A 251, 233 (1959) ; W. Bonnor and M. Rotenberg, Proc. R. Soc. London, Ser. A 289, 247 (1966) ; K. S. Thorne, Rev. Mod. Phys. **52**, 299 (1980) ; L. Blanchet and T. Damour, Phys. Rev. D **37**, 1410 (1988) ; L. Blanchet and T. Damour, Phys. Rev. D **46**, 4304 (1992) ; L. Blanchet and G. Schaefer, Class. Quant. Grav. **10**, 2699-2721 (1993).

[137] T. Damour and P. Rettegno, Phys. Rev. D **107**, no.6, 064051 (2023) [arXiv :2211.01399 [gr-qc]].

[138] M. Khalil, A. Buonanno, J. Steinhoff and J. Vines, Phys. Rev. D **106**, no.2, 024042 (2022) [arXiv :2204.05047 [gr-qc]].

[139] T. Damour, F. Guercilena, I. Hinder, S. Hopper, A. Nagar and L. Rezzolla, Phys. Rev. D **89**, no.8, 081503 (2014) [arXiv :1402.7307 [gr-qc]].